On absolute (robust) stability: slope restrictions and stability multipliers

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SUMMARY

Absolute stability for systems with several sector-restricted and slope-restricted nonlinearities is studied in this paper. A critical analysis of the multipliers is performed and the multipliers of Yakubovich type are chosen because the stability inequality is obtained with a minimum of technical assumptions. The main part of the paper is devoted to obtaining the Yakubovich-type criterion in the unified context of stable, critical, and unstable cases for the linear part. The paper is motivated by the problem of pilot in-the-loop oscillations of the aircrafts where critical and unstable cases appear and the saturation nonlinearity is both sector and slope restricted. The paper contains some applications of the frequency domain inequalities. The conclusions show a ‘parsimony principle’: using as few free parameters as possible to obtain the largest possible domain of stability. The paper ends with conclusions and hints for further development. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION: A HISTORICAL PERSPECTIVE

The first studies in this field (the so-called absolute stability) were performed some 70 years ago.‡ Absolute stability deals with global asymptotic stability of the equilibrium at the origin for the feedback structure of the type shown in Figure 1. In this structure, the forward path contains a dynamic linear-time invariant system, whereas the feedback path contains a memoryless (possibly time-varying) nonlinear element. This element represents, in fact, a class of such elements because the only information about the nonlinearity is its membership to a certain class. Consequently, the property of global asymptotic stability is the property of every system corresponding to some specific nonlinearity of the considered class. With respect to this, absolute stability provides conditions for robust stability with respect to a given class of uncertain elements. It is worth mentioning here that the pioneers of the field considered this aspect—see the early report [5]. Their terminology at that time was ‘poor information about the nonlinear element’ (see also [6]).

The literature on absolute stability is extensive: thousands of papers have been published during several decades and many books have been written. The peak of the research was reported in the 1960s. The interest for the problem is however renewed because of various applications. The connections with such ‘hot’ areas of the control theory research as linear quadratic optimization including $H_{\infty}$, robustness, stability radii, passivity, and dissipativeness of systems should be also mentioned (see [7–9]).

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‡Unlike most of the people in the field, we consider [1] and not [2], the paper that announces the absolute stability problem. The reader may also consult [3,4].

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There are several criteria to distinguish the results of the field: the class of allowable nonlinear elements, the class of allowable linear elements, and the method applied in obtaining sufficient conditions for absolute stability. In fact, all these aspects are strongly connected, as it will be shown.

For the feedback nonlinear elements, the small gain and circle theorems guarantee absolute stability for arbitrarily time-varying (but sector-restricted) nonlinearities; in [8], it is shown that this is not the case with the Popov criterion. In fact, the extended Popov criterion [10] corresponds, if the Popov proportional derivative (PD)-type multiplier is reduced to a proportional-type one, to the circle criterion. Of course, these are frequency domain stability criteria, and according to Yakubovich–Kalman–Popov lemma, each of them corresponds to a certain Liapunov function. Thus, the circle criterion corresponds to a fixed quadratic form, whereas the Popov criterion with PD multiplier corresponds to a ‘quadratic form plus integral of the nonlinearity’, the integral being path independent only for time-independent nonlinearities. This leads us to the problem of the methods.

The oldest method in absolute stability is the Liapunov method; it is suitable mainly for finite-dimensional systems, and its abstract extensions require caution in applications. The original criterion of Popov [11] as well as the circle criterion are based on input/output methods (integral relations and Fourier–Plancherel theory). It is this approach that allowed the results’ extensions to systems whose linear blocks have infinite dimension (time lag or distributed parameters [12, 13]). In fact, as pointed in [14], there exists a competition of the two methods, each of it having its advantages and shortcomings.

Less known is the way in which these methods interact. The input/output method is based on constructing an integral index incorporating information about the nonlinearity; this was pointed out in [15]. Up to this paper, the index construction looked somehow dictated by inspiration (the ‘a priori’ integral). Afterwards, one has to obtain the basic stability inequality; this can be performed either by Fourier–Plancherel theory or by applying Yakubovich–Kalman–Popov lemma to obtain a Liapunov function with the structure ‘quadratic form plus (possibly) an integral of the nonlinearity’. The way of interaction and competition of the two methods is well illustrated by the effort of extending the absolute stability criteria to slope-restricted or monotonic nonlinearities. For monotonic or odd monotonic nonlinearities, the best-known references are the papers [16–20]. The criteria could be obtained by using the extended Liapunov functions, but the most reasonable way seems to be that of an extended linear system and of a properly chosen integral index incorporating available information about nonlinearity.

Here, the reference to the more recent but widely cited papers on integral quadratic constraints (IQC in e.g., [21] and its references) is useful. There are two underlying ideas, and, as usually, they deal with the two subsystems of Figure 1. The first one arises from an extension of [15]: the information on the nonlinear element was incorporated in various quadratic constraints on subsystem’s input and output signals. Because [15] contains a list of such constraints, the IQC approach aimed to extend this list by discovering new ‘hidden’ information about the nonlinearities. Moreover, even the time delay in the linear part was considered ‘unpleasant’ and described by some approximating IQCs. This idea on time delay appeared for the first time in an early paper, which, by now, is almost forgotten [22] (among other results, this paper announces the hyperstability theory). With such option, the linear part remained time invariant and with lumped parameters, that is, described by a rational (matrix) transfer function. In this way, not only the frequency domain inequality but also the positivity theorem (Yakubovich–Kalman–Popov lemma) may be applied in the finite-dimensional case. This sends us to a Liapunov function defined on finite-dimensional spaces and finally to linear
matrix inequalities. As follows from [21], this has a corresponding connection with the choice of the multipliers in the frequency domain inequality.

The discussion on multipliers is interesting in its turn: the first multiplier is the PD multiplier of [11, 23] that reads $\zeta(\sigma) = 1 + \beta\sigma$. It is worth mentioning that various multipliers were introduced when considering monotone and slope-restricted nonlinear functions [16–18]. The more recent papers [24, 25] gather all these cases under a single label—systems with extended state space—and constructs a framework incorporating even noncausal (anticipative) subsystems. The idea has been circulated, independently, in [26] and [27] (these results remained without sensible impact, being nevertheless cited in [10]).

Some other researches [28–32] reported a new absolute stability criterion for systems with locally slope-restricted nonlinearities. Two were the claims for the new results. The first claim was that the criterion incorporated only slope information about the nonlinear elements. Further, the Popov PD multiplier $\zeta(\sigma) = 1 + \beta\sigma$ was replaced by a new ‘proportional integral (PI)’ multiplier $1 + \beta\sigma^{-1}$. It requires the function $(1 + \beta\sigma^{-1})(k^{-1} + \chi(\sigma))$ to be positive real, where $\chi(\sigma)$ represents the transfer function of the linear dynamic element and $k$ is a bound on the local slope of the feedback nonlinearity. As pointed out in [31], the predictions of the new criterion were less conservative than those of the standard Popov criterion in those cases when the Nyquist plot of $\chi(\sigma)$ resided in the first and second quadrants. One might see this as an advantage of the proposed criterion.

The PI multiplier also has its history. Within the framework of the integral equations, it appeared for the first time in [33] (see also [34]) and later in a paper of V. M. Popov himself [35]. For systems described by ordinary differential equations, the same multiplier occurred in [36].

There are, however, some restrictions that have to be taken into account for this inequality, restrictions arising from the proof technique. As a matter of fact, these proofs are quite involved. It is thus not surprising that some omissions occurred. For instance, in [28], an extended system is associated by differentiating the initial one. The Liapunov function considered in that paper could have the required properties only by restricting the extended system to an invariant set where the solutions of the basic system belong [30]. As a consequence, in the multivariable case, the frequency domain stability inequality of [30] required that the matrix transfer function $H(\sigma)$ of the linear subsystem should satisfy $H(0) > 0$ and that $H(0)$ should be diagonal. Because of the framework of [30] being quite general, the authors of [31] turned to the basic case of a stable linear subsystem with the slope restriction $0 < \phi'(v) < k$. They also associated a new system by differentiation as in [28] but used the so-called kinetic Liapunov function [37].

This analysis will be detailed in the next section. Here, we add only the fact that the result of [35], obtained via the input/output approach, which is applied to integral equations, included both the cases $\chi(0) > 0$ and $\chi(0) < 0$. The ‘pay-off’ is reflected in the conditions on the nonlinearity.

About the notations. The notations are an important option in the field of the absolute stability because they should allow us to distinguish among the single variable and multivariable cases (single or several nonlinear elements) as well as among scalars, vectors, and matrices. For this reason, we shall follow throughout this paper the main notation convention of the field, due especially to V. A. Yakubovich: the lowercase Greek letters account for scalars (except the independent—‘time’—variable, which is denoted, as usual, by $t$), lowercase Latin letters account for vectors, and capital Latin letters account for matrices. We shall use the calligraphic (‘mathcal’ set of) letters for the elements of the Popov system ‘differential equations plus integral index’. In order to avoid confusion, we will explain all deviations from the aforementioned convention.

2. MULTIPLIERS AND AUGMENTED SYSTEMS

The discussion of the previous section was made in order to emphasize that we deal, in fact, with the already mentioned extended state space. The way of introducing this extension is not without importance because it induces the proof and the technical assumptions associated with it. To be more specific, we start from the remark that the frequency domain condition of [30, 31] may be obtained from the earlier condition derived in [19, 20]. Assume that

$$0 < \phi'(v)/\nu < \mu_0 \quad , \quad \nu < \phi'(v) < \nu'$$

(1)
and the matrix $A$ of the system
\[ \dot{x} = Ax - b\varphi(c^*x) \] (2)
is a Hurwitz matrix. The frequency domain stability inequality is
\[ \tau_1 \left( \frac{1}{\mu_0} + \Im \chi(i\omega) \right) + \tau_2 \Im e^{i\omega}(i\omega) + \tau_3 \omega^2 \Im e^{(1 + \Im \chi(i\omega))(1 + \Im \chi(i\omega))} \geq 0 \] (3)
for some real numbers $\tau_i$, where $\tau_1 \geq 0$, $\tau_3 \geq 0$, $\chi(\sigma) = c^*(\sigma I - A)^{-1}b$ being the transfer function of the linear part. If we adopt the viewpoint of [28, 31] to take into account only the information about the slope, then $\tau_1 = 0$. By multiplying with $\omega^{-2}$ and choosing $\tau_3 = \Im ^{-1}$, it follows
\[ \frac{1}{\Im} + \Im e \left( 1 + \frac{\Im}{\Im} - \frac{\tau_2}{\omega} \right) \chi(i\omega) + \left( \frac{\Im}{\Im} \right) |\chi(i\omega)|^2 \geq 0, \] (4)
and if $\Im = 0$ in the slope restrictions, the criterion with the PI multiplier is obtained.

On the other hand, the same condition (4) is obtained in [30] within a more general framework but using essentially the same extended state space as in [28], together with some additional assumptions. One of them is as follows. For the system with several nonlinear elements
\[ \dot{x} = Ax - \sum_{k=1}^{m} b_k \varphi_k(c_k^*x), \] (5)
the matrix transfer function $H(\sigma)$ with its entries $\chi_{kl}(\sigma) = c_k^*(\sigma I - A)^{-1}b_l$ should have the so-called static decoupling property: $\chi_{kl}(0) = 0$, $k \neq l$. It is interesting to notice that if the approach of [35] is extended to systems described by vector integral equations, the same static decoupling property is required.

A possible reasonable explanation is as follows: there exists some connection between the association of the extended system and the proof technicalities. In order to give a brief illustration of this statement, we shall turn to the case of a single nonlinear element described by (1). In [19, 20], the following extended state variables are considered:
\[ z = x, \quad \zeta = -\varphi(c^*x) \] (6)
which sends us to the $(n+1)$-dimensional system
\[ \dot{z} = Az + b\zeta, \quad \dot{\zeta} = -\varphi'(c^*z)c^*(Az + b\zeta). \] (7)

It is now clear that this system has a prime integral
\[ \zeta(t) + \varphi(c^*z(t)) \equiv \text{const}. \] (8)
Hence, its dimension may be reduced by 1. If the solutions of (7) are viewed on the previously defined invariant set, then $z(t) \equiv x(t)$ provided $z(0) = x(0)$. This extended system is considered in [38] for the case of several nonlinear functions.

We now turn to the approach of [36], which is the second, chronologically speaking. Here, the state variables are
\[ z = Ax - b\varphi(c^*x), \quad \zeta = -\varphi(c^*x), \] (9)
which shows that both definitions contain nonlinear relations. The $(n+1)$-dimensional system will be
\[ \dot{z} = Az + b\zeta, \quad \dot{\zeta} = \eta, \quad \eta = -\varphi'(c^*x)c^*z, \] (10)
but $x(t)$ from the linear basic system is still present. One might further associate a linear system with restricted time-varying gain

$$
\dot{z} = Az + b\zeta,
\dot{\zeta} = -\lambda(t)c^*z, \quad \forall < \lambda(t) < \forall,
$$

but this will hide the genuine nature of the system. The things become simpler if $\det A \neq 0$ (this holds not only for $A$, a Hurwitz matrix, but also for hyperbolic matrices as well as for all critical cases with nonzero eigenvalues). One may compute $c^*x = c^*A^{-1}z - c^*A^{-1}b\zeta$ and obtain the system

$$
\dot{z} = Az + b\zeta,
\dot{\zeta} = -\varphi'(c^*A^{-1}z - c^*A^{-1}b\zeta)c^*z,
$$

which has the prime integral

$$
\zeta(t) + \varphi(c^*A^{-1}z(t) - c^*A^{-1}b\zeta(t)) \equiv \text{const}
$$

The third approach is that based on ‘differentiation’ [28], with its multivariable version [30, 31]. The new state variables are

$$
z = Ax - b\varphi(c^*x), \quad \zeta = c^*x,
$$

which satisfy

$$
\dot{z} = Az - b\varphi'(\zeta)c^*z,
\dot{\zeta} = c^*z.
$$

This system has the prime integral

$$
\zeta(t) - c^*A^{-1}z(t) - c^*A^{-1}b\varphi(\zeta(t)) \equiv \text{const}.
$$

Not only the use of (6) is an easier way of defining the new state variables but also the ‘return’ to the basic system (1) via the associated prime integral, which generates a family of invariant sets, is much simpler.

An interesting discussion is to be presented here. All auxiliary systems introduced by (10), (12), and (15) contain the derivative of the nonlinear function. It is exactly this feature that allows the introduction of the slope restrictions, as it will appear in the next sections. On the other hand, in two quite recent papers [39, 40], the differentiated (even in a generalized sense) nonlinearity is introduced from the very beginning. More precisely, the absolute stability problem is considered for (possibly) discontinuous nonlinear functions that are also monotone. The structure of the system is, in [39, 40], as follows:

$$
\dot{x} = Ax - b\mu_L,
\mu_L \in \partial\varphi(v), \quad v = c^*x,
$$

with $\partial\varphi$—the subdifferential—a natural option for nonsmooth functions. This kind of systems, which is well motivated in [39, 40], is connected to some pioneering papers of the 1960s that are from A. K. Gelig [41–43], where bounded and discontinuous nonlinear functions are considered. The approach is, however, different. Whereas the approach in [39, 40] uses the theory of maximally monotone operators, the approach in [41–43] is within the framework of the classical absolute stability theory. An important problem has been the sense of solution for the systems with discontinuous right-hand side. It is not the case to give a detailed explanation, but the approach of [43] started from the Filippov sense, and it turned out that the Filippov sense was too restrictive whereas other definitions (e.g., Carathéodory, Krasovskii, Aizerman, and Gantmakher) were too weak. Finally the problem has been studied in some detail in [44]. The system form of (17) is much alike to that of [44] in the sense that $\mu_L$ is defined from the multifunction of the inclusion and that $x(t)$ from the
linear system as the absolutely continuous (Carathéodory) solution of the forced system. This definition of the solution is very suited for the absolute stability problem, and what remains of the paper will strengthen this assertion. Moreover, this approach incorporates in a natural way the problem of the sliding modes [44].

The comparison with [39, 40] motivates the discussion of the so-called feedthrough term in the argument of the nonlinear function. If the form (17), that is, the framework of the multifunctions, is adopted, then one may take \( v = c^*x + \gamma \mu_L \) and construct the solution with the only problem of overcoming the presence of \( \mu_L \) in both sides of the inclusion relation. Consideration of the feedthrough term also has its roots in the classical absolute stability theory—the so-called tachometric feedback [45]. Because ‘tachometric’ meant ‘velocity feedback’, the corresponding system was as follows:

\[
\dot{x} = Ax - b \varphi(v),
\]

\[
v = c_0^* x + \gamma (c_1^* \dot{x}),
\]

and one can see a feedthrough term that is the derivative of a certain output. We deduce further:

\[
v + \gamma (c_1^* b) \varphi(v) = (c_0^* - \gamma c_1^* A)x.
\]

Here, an interesting discussion may be carried out. If \( c_1^* b = 0 \), then the case is trivially reduced to the standard one. This condition is not so restrictive as one might think: if the transfer function \( c_1^* (\sigma I - A)^{-1} b \) is considered, \( c_1^* b \) is its first Markov parameter and \( c_1^* b = 0 \) means nothing more but a relative degree at least 2. If \( \gamma (c_1^* b) > 0 \) and \( \varphi \) is monotone, then the function \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( \psi(v) = v + \gamma (c_1^* b) \varphi(v) \) is invertible, and we may write

\[
\dot{x} = Ax - b \varphi(\psi^{-1}(v)),
\]

\[
v = (c_0^* - \gamma c_1^* A)x,
\]

and this is a standard absolute stability problem, provided \( \varphi \circ \psi^{-1} \) is subject to some corresponding sector and/or slope restrictions.

Finally, the discontinuities of the nonlinear functions in [39, 40] introduce semi-infinite sector and/or slope restrictions on the nonlinear functions. This is not surprising because the pioneering papers [1, 2] displayed the condition \( \varphi(v)v \geq 0 \)—a nonlinear function confined to the first and third quadrants, that is, with infinite upper bound of the sector. The semi-infinite sectors, although addressed in the classical absolute stability theory from the beginning, require some caution in manipulating both Liapunov functions and frequency domain inequalities. This aspect will be addressed in the sequel.

In the following, we shall focus on the system (7) and undertake a generalization to the multivariable case—a step forward in comparison with [38] as well as with other papers dealing with the multivariable case that followed it. That is, a unified treatment of stable, critical, and unstable cases will be considered (in the line of [10, Chapter 5]). Also, both sector and slope restrictions will be considered.

### 3. THE MAIN RESULT AND SOME APPLICATIONS

#### 3.1. The result

In the following, we shall consider the system

\[
\dot{x} = Ax - \sum_{i=1}^{m} b_i \varphi_i(c_i^* x)
\]

with \( \text{dim } x = \dim b_i = n \) \((i = 1, n)\). When necessary, we shall denote by \( B \) and \( C \) the \( m \times m \) matrices having columns as columns \( b_i \) and \( c_i \), respectively. The \( C^1 \) functions \( \varphi_i : \mathbb{R} \rightarrow \mathbb{R} \) are subject to the sector and slope restrictions

\[
\varphi_i' \leq \varphi(v)/v \leq \varphi_i'' \, , \, \varphi_i' \leq \varphi'(v) \leq \varphi_i'' \, , \, i = 1, m.
\]
SLOPE RESTRICTIONS AND STABILITY MULTIPLIERS

It is an elementary fact that one may assume that \([\underline{\varphi}^i, \overline{\varphi}^i] \subseteq [\underline{\nu}^i, \overline{\nu}^i]\) without restricting the admissible class of nonlinear functions [19].

Definition 1
Following [10], we shall call system (23) minimally stable if there exists a set of numbers \(\overline{\varphi}_i \in [\underline{\varphi}^i, \overline{\varphi}^i] \subseteq [\underline{\nu}^i, \overline{\nu}^i], i = 1, m,\) such that the matrix \(A - \sum_1^m b_i \overline{\varphi}_i c^*_i\) has all its roots in \(\mathbb{C}^-\) (is a Hurwitz matrix).

The significance of the notion is obvious: if we want to achieve global asymptotic stability for all nonlinear (and linear) functions subject to (24), this property should hold at least for one linear function.

In the statement of the main result, we shall make use of the following sets of nonnegative free parameters: \(\tau_i \geq 0, \theta_i^I \geq 0, \theta_i^{II} \geq 0, \beta_i \geq 0, i = 1, m;\) we denote by \(D_1, D_2', D_2'', D_3\) the \(m \times m\) diagonal matrices with diagonal elements \(\tau_i, \theta_i^I, \theta_i^{II}, \beta_i\) respectively. Also, let \(\underline{\varphi}, \overline{\varphi}, \underline{\nu}, \overline{\nu}\) be the \(m \times m\) diagonal matrices having as diagonal elements the sector and slope restrictions \(\varphi^i, \overline{\varphi}^i, \nu^i, \overline{\nu}^i, \) respectively.

Let \(H : \mathbb{C} \mapsto \mathbb{C}^{m \times m}\) be the matrix transfer function of the linear system
\[
\dot{x} = Ax + Bu(t), \ y = C^*x
\]
with the entries
\[
\chi_{jk}(\sigma) = c_j^*(\sigma I - A)^{-1}b_k, \ j, k = 1, m
\]
and expressed, for short, as \(H(\sigma) = C^*(\sigma I - A)^{-1}B.\) We may state now the main theoretical result of the paper.

Theorem 1
Consider system (23) under the following basic assumptions: (i) it is nondegenerate, that is, \((A, B)\) is a controllable pair and \((C^*, A)\) is an observable pair; (ii) it is minimally stable in the sense of Definition 1; and (iii) the \(\varphi^i\)-functions \(\varphi_i : \mathbb{R} \mapsto \mathbb{R}\) are such that \(\varphi_i(0) = 0\) and subject to the sector and slope restrictions (24).

Suppose there exist real nonnegative numbers \(\tau_i \geq 0\) and \(\beta_i \geq 0\) and the real numbers \(\theta_i, i = 1, m,\) such that the following frequency domain matrix inequality holds for all \(\omega \geq 0:\)
\[
T(\omega) = \left(D_1 \overline{F}^{-1} + \omega^2 D_3 \overline{G}^{-1}\right) + \Re \left(D_1 \left(1 + \overline{F} \overline{F}^{-1}\right) + \omega D_2 + \omega^2 D_3 \left(1 + \overline{G} \overline{G}^{-1}\right)\right) H(\omega)
\]
\[
+ H^*(-\omega) \left(D_1 F + \omega^2 D_2 G\right) H(\omega) \geq 0,
\]
where matrices have been defined except for \(D_2\), which is the diagonal matrix with \(\theta_i\) as diagonal entries. Moreover, the following alternative is valid.

If \(A\) is hyperbolic, that is, has no eigenvalues on the imaginary axis \(i\mathbb{R}\), then (27) is strict (including \(\omega \to \infty\)), whereas (24) are nonstrict; if \(A\) has some eigenvalues on \(i\mathbb{R}\), that is, system (23) is in a critical case, then (27) is nonstrict, whereas (24) are strict; if, in particular, \(T(0) = 0\), then either \(\sigma = 0\) is not an eigenvalue of \(A\), that is, \(\det A \neq 0\), and in this case, the following matrix inequality holds:
\[
\frac{1}{2} \left(\overline{F}^{-1} + H^*(0)\right) (I + \overline{F} H(0)) + \frac{1}{2} (I + H^*(0) F) (\overline{F}^{-1} + H(0)) > 0,
\]
or \(\sigma = 0\) is an eigenvalue of \(A\), that is, \(\det A = 0\), then \(\overline{F} \overline{F} > 0\), that is, \(\underline{\varphi}^i \overline{\varphi}^i > 0\) for all \(j = 1, m,\)

Assume also that the following conditions are fulfilled:
\[
\liminf_{\lambda \to \infty} \frac{\theta_j}{\lambda} \left(\int_0^\lambda \varphi_j(\eta) d\eta - \frac{1}{2} \lambda \varphi_j(\lambda)\right) \geq 0, \ j = 1, m.
\]
Then system (23) is absolutely stable, that is, its equilibrium at 0 is globally asymptotically stable for all nonlinear functions in the corresponding admissible class.

We shall not comment here on all the features of this result because a comparison has already been performed in the previous sections. We will just mention the unified treatment of what were called, in the classical absolute stability theory, stable, critical, and unstable cases of the linear part, the elimination of the static decoupling condition of [30] \( (H(0) \text{ need no longer be diagonal}) \), and unlike in [30], the admissible class of nonlinear functions is no longer dependent on the numbers \( \varphi_j \) defining the minimal stability. On the other hand, this class of criteria does not meet the requirement stated in [28] and followed in [30–32] to take into account only slope restrictions. This would require taking \( D_1 = 0 \) and \( D_2 = 0 \), and consequently, the criterion would be just another one of the circle type, without freely chosen parameters. It appears that incorporation of the information on nonlinearity (sector restrictions, slope restrictions, monotonicity, parity) in one or in other combination might be in connection with the way of associating the augmented system. With respect to the specific problem of incorporating sector restrictions together with slope restrictions or slope restrictions only, the augmented system defined by (14) might be better suited for slope restrictions only than the system defined by (6). A deeper insight on these aspects could be obtained by a thorough comparative analysis of the corresponding references [20, 25, 30, 31, 36].

It is also useful to comment the ‘limit’ cases of the frequency domain inequality that arise from the alternative induced by the location of the eigenvalues of \( A \), an alternative that is clearly stated in [10] as basic in absolute stability. In the hyperbolic case of \( A \) (no eigenvalues on \( i\Re \)), (27) must be strict including \( \omega \to \infty \). Because \( H(\sigma) = C^*(\sigma I - A)^{-1} B \), the matrix transfer function of the linear part of (23) is strictly proper, \( \Re \{ H(i\omega) \} = \partial(|\omega|^{-1}) \) and approaches 0 for \( \omega \to \infty \). This sends us to the fact that only proper transfer functions (with relative degree 0) may be strictly positive real. But, in our case, it is \( T(i\omega) \) that has to be strictly positive for \( \omega \to \infty \) including. For the Popov-like criterion, the terms \( D_1 F^{-1} \) and \( i\omega D_2 H(i\omega) \) are helpful, whereas for (27), we have also the terms containing \( \omega^2 D_3 H(i\omega) \) and especially \( \omega^2 D_3 G^{-1} \).

These conditions send us, however, to another problem that has been already mentioned at the end of Section 2—the semi-infinite sector and slope restrictions. If a sector is semi-infinite, the corresponding entry in \( F^{-1} \) becomes 0, and because \( [\varphi^j, \varphi^l] \subseteq [u^j, u^l] \), the corresponding entry in \( G^{-1} \) also becomes 0. We deduce that what is left of (27) must be positive definite for \( \omega \to \infty \) to ‘help’ those terms that are only nonnegative definite. In fact, the choice of \( D_1, D_2, \) and \( D_3 \) should allow fulfillment of the strict (including \( \omega \to \infty \)) inequality (27).

The condition (28) that appears in the second case of the alternative, when (27) is nonstrict and (24) are strict, has a rather technical character that will be commented in due time. We may, however, consider here a more significant interpretation of this condition. If \( T(0) = 0 \), this does not contradict the nonstrict inequality (27) but would not be helpful at some stage of the proof. Now, the matrix \( D_1 \) is multiplied in \( T(i\omega) \) by some matrix frequency domain function that we require to be strictly positive at \( \omega = 0 \). It reads

\[
\overline{F}^{-1} + \frac{1}{2}(I + \overline{F}F^{-1})H(0) + \frac{1}{2} H^*(0)(I + \overline{F}F^{-1}) + H^*(0)FH(0) > 0,
\]

which is immediately factored as (28). The scalar version of (28) looks as

\[
(1 + \varphi \chi(0)) \left( \frac{1}{\varphi} + \chi(0) \right) > 0,
\]

which contains some information about the starting point of the Nyquist locus. We may consider (28) as the multivariable version of it. As already pointed out, the technical usefulness of (28) will appear in due time.

Condition \( \varphi \chi/\chi' > 0 \) reproduces in the multivariable case the fact that, in critical and unstable cases, the identically zero linear characteristic is not included in the absolute stability sector; here, this assumption is again technical as it will appear in what follows.
Condition (29) has a technical character also. It appeared for the first time in a paper of Yakubovich from 1962, presented at an all-union (Union of Soviet Socialist Republics) conference on applied mathematics and mechanics in Kazan; it can be found afterwards in the papers of Yakubovich [19, 20] as well as in [38]. Following the comments of [19, 20], we can give (29) a significance that is not only technical. It is a condition for the behavior of \( \psi(v) \) at large deviations—a necessary condition to obtain global asymptotic stability from a Liapunov function (in particular, in our case, we need this condition in order to make use of Lemma 1 [see the end of Section 4]).

This behavior at large deviations does not follow in a necessary way from the sector and/or slope restrictions. With respect to this, it is interesting to recall here the second-order counterexample of N. N. Krasovskii to the conjecture of Aizerman. The counterexample is from 1952 and reads

\[
\dot{\xi} = \xi + \eta - \psi(\xi), \quad \dot{\eta} = -\xi - \eta, \tag{30}
\]

where

\[
\psi(\xi) = \begin{cases} 
e^{-\xi}(1 + e^{-\xi})^{-1}, & \xi \geq 1, \\ e^{-\xi}(1 + e^{-1})\xi, & \xi < 1, \end{cases} \tag{31}
\]

and it is easily seen that \( \psi(v) > 0 \). If \( \psi(v) = \alpha v \), the Hurwitz condition will give \( \alpha > 0 \). One might hope, as known, that the conjecture of Aizerman holds for this system as for any second-order system. However, the solution corresponding to \( \xi(0) = 1 \) and \( \eta(0) = e^{-1} - 1 \) has the property that \( \lim_{t \to \infty} \xi(t) = +\infty \). On the other hand, it is seen from (31) that

\[
\lim_{v \to -\infty} \psi(v) = -\infty, \quad \lim_{v \to \infty} \psi(v) = 0.
\]

Also,

\[
\lim_{v \to \pm\infty} \int_{0}^{v} \psi(\lambda)d\lambda = +\infty, \quad 0 < \lim_{v \to \pm\infty} \int_{0}^{v} \psi(\lambda)d\lambda < +\infty.
\]

In order for us to ‘save’ the Aizerman conjecture for second-order systems, the assumption

\[
\lim_{v \to \pm\infty} \int_{0}^{v} \psi(\lambda)d\lambda = +\infty \tag{32}
\]

was added. The example of Krasovskii clearly shows that (32) does not follow from the sector condition \( \psi(v) > 0 \).

Coming back to [19, 20], we find that (29) holds if either

\[
\lim_{|v| \to \infty} \frac{1}{v^2} \left[ \int_{0}^{v} \psi(\lambda)d\lambda - \frac{1}{2} v\psi(v) \right] = 0,
\]

or there exists some \( \nu_* > 0 \) such that

\[
\theta \left[ \int_{0}^{v} \psi(\lambda)d\lambda - \frac{1}{2} v\psi(v) \right] \geq 0, \quad |v| \geq \nu_*,
\]

or there exist the finite limits \( \lim_{v \to \pm\infty} \psi(v)/v \).

Consider again the function defined by (31). A quite elementary manipulation shows that the first of the three aforementioned conditions is fulfilled. Now, let \( \psi(v) = \gamma v^3, \gamma > 0 \). Another straightforward manipulation shows that (29) does not hold. It appears that (29) ‘moderates’ the growth of the functions \( \psi(v) \) for large deviations. In particular, the case of the cubic parabola \( \gamma v^3 \) shows that (29) is effective mainly for semi-infinite sectors and slopes.

In the following, we shall present some applications of the theorem. As a general characteristic of our approach, we shall obey some kind of ‘parsimony’ principle: to have as few free parameters as possible in the frequency domain inequality for recovering as much as possible from the linear stability sector for the nonlinear one (reducing the ‘conservatism’, that is, the gap between sufficient and necessary and sufficient conditions for stability in the nonlinear case).
3.2. Applications to cases with a single nonlinear element

Such applications having a purely mathematical character may be found in [38, 46] or even in earlier papers [47, 48].

We return first to the celebrated counterexample of Pliss [49] that disproved the Aizerman conjecture, not for some relaxation of the already well-known stability conditions, for example, [30, 35], but because it may illustrate some principles, being at the same time nonstandard. The transfer function of the linear part is

$$\chi(\sigma) = \frac{1}{\sigma + 1 + 1} + \frac{\sigma - 1}{\sigma^2 + 1} \quad (a > 0).$$  (33)

Because the matrix transfer function $H(\sigma)$ reduces here to a scalar, we changed the notation in $\chi(\sigma)$, using a lowercase Greek letter as announced. Its irreducibility is easily checked; hence, the linear part is controllable and observable; also the system is in the critical case of two imaginary eigenvalues (poles).

The characteristic equation of the linearized system

$$1 + \gamma \chi(\sigma) = 0$$

allows determination of the maximal (Hurwitz) sector of stability, namely $0 < \gamma < (1 + a)/a$; this shows that the maximal achievable result is subject to $\varphi \geq \nu > 0$ and $\overline{\varphi} \leq \overline{\nu} < (1 + a)/a$.

Application of the Popov criterion requires here $\frac{1}{\varphi} > 0$, $\frac{1}{\varphi} > 0$, and also $\varphi = 0$, $\nu = 0$, $\overline{\nu} = 0$, it follows that the frequency domain inequality (27) becomes

$$\frac{1}{\varphi} + \Re(e(1 + \i \omega \theta)\chi(t \omega)) \geq 0.$$  

This reads

$$\frac{1}{\varphi} + \frac{(1 + a)(1 + \omega^2 \beta) + \theta \omega^2}{(1 + a)^2 + \omega^2} - \frac{1 + (\beta + \theta)\omega^2}{1 - \omega^2} \geq 0,$$

and a necessary choice is $\theta = -1 - \beta$ because $A$ has two conjugate purely imaginary eigenvalues. This gives

$$\frac{1}{\varphi} + \frac{(a \beta - 2)\omega^2 - a(1 + a) \omega^2}{\omega^2 + (1 + a)^2} \geq 0, \quad \forall \omega \geq 0.$$  

A short discussion shows that the best choice for $\beta$ would be $\beta > 2/a - (1 + a)^{-1}$. This gives the fulfillment of the aforementioned inequality for $\overline{\varphi} < (1 + a)/a$, recovering thus the entire Hurwitz sector, as already known.

We shall consider now the stability problem for the roll attitude control system for a fighter aircraft [50]. As mentioned in the cited reference, ‘the lateral bare airframe dynamics were modeled using simple roll-subsidence-only approximations. The primary variables were the roll-mode time constant $T_R$ and the roll control sensitivity $L_{\delta_y}$. . . . The lateral Dutch roll mode was well damped and was, therefore, not a factor in the study’. The flight control system (FCS), which acts also as
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stability augmentation system (SAS), is represented by roll-subsidence angle and roll rate signals that are fed back via an actuator that is also a rate limiter. The pilot model is the so-called compliant pilot—a simple gain—ensuring an additional gain for the angle feedback. Consider the following differential equations:

\[
\ddot{\phi} + \frac{1}{T_R} \dot{\phi} = L_{\delta_a} \delta_a \\
\delta_a = \omega_e \psi(\phi - k_p \dot{\phi} - \delta_a)
\]  
\[(34)\]

with \(\phi\), the roll-subsidence angle, and \(\delta_a\), the control signal (the actuator output); \(k_\phi\) is the roll angle feedback gain, \(k_p\) is the roll rate feedback gain, and \(k_H\) is the human (compliant pilot) gain. The nonlinear function \(\psi(\nu)\) is the saturation function having the slope equal to \(V_L/\epsilon_L\) in the linear (unsaturated) zone, \(V_L\) being the saturation level and \(\epsilon_L\) the input saturation level.

In the linear case, when \(\psi(\nu) \equiv \nu\), the characteristic equation reads

\[
\sigma^3 + (\omega'_e + \frac{1}{T_R}) \sigma^2 + \omega'_e (\frac{1}{T_R} + k_p L_{\delta_a}) \sigma + \omega'_e L_{\delta_a} (k_\phi + k_H) = 0, 
\]  
\[(35)\]

and all its coefficients are strictly positive. Here, \(\omega'_e = \omega_e V_L/\epsilon_L\). The Routh Hurwitz condition reads finally as

\[
k_\phi + k_H < \left(\omega'_e + \frac{1}{T_R}\right) \left(k_p + \frac{1}{T_R L_{\delta_a}}\right),
\]  
\[(36)\]

showing an obvious stability limitation over the global roll angle gain, that is, also on the pilot gain; the helpful role of the roll rate gain \(k_p\), as well as of a fast actuator (the factor \(\omega_e\)), is also quite clear.

Consider now the rate limiter being active. The equations (34) display an absolute stability structure as follows:

\[
\ddot{\phi} + \frac{1}{T_R} \dot{\phi} = L_{\delta_a} \delta_a, \quad \dot{\delta}_a = \omega_e \mu, \\
v = -(k_\phi + k_H) \phi - k_p \dot{\phi} - \delta_a; \quad \mu = -\psi(\nu).
\]  
\[(37)\]

We deduce the transfer function of the linear block—again denoted by the Greek letter \(\chi(\sigma)\) instead of \(H(\sigma)\) as it was in the multivariable case.

\[
\chi(\sigma) = \frac{\tilde{\nu}(\sigma)}{\mu(\sigma)} = \frac{\omega_e}{\sigma} \left(1 + L_{\delta_a} k_\phi + k_H + k_p \sigma\right),
\]  
\[(38)\]

which clearly shows the critical case of a double zero root with nonsimple elementary divisors. This case was recognized as quite special and ‘very critical’ (see, e.g., [12]); even V. M. Popov considered this case in a distinct paper [51]. The case is nevertheless incorporated in the general Theorem 1. Because of the nonlinearity, we have clearly \(\nu = 0, \phi = 0\). Consider first the Popov criterion, that is, \(\beta = 0\). It reads as follows:

\[
\frac{\tau}{\phi} + \omega_e \mu e(\tau + i \omega \theta) \left(\frac{1}{i \omega} - L_{\delta_a} \frac{k_\phi + k_H + i \omega k_p}{\omega^2(i \omega + 1/T_R)}\right) > 0;
\]

hence,

\[
\frac{\tau}{\omega_e \phi} + \theta - \frac{L_{\delta_a}}{\omega^2(\omega^2 + 1/T_R)}(\tau/T_R + \omega^2 \theta)(k_\phi + k_H + (\tau - \theta/T_R)k_p \omega^2) > 0.
\]

For an obvious change of sign at some \(\omega > 0\) to be avoided, the only choice is \(\tau = 0\) (the ‘infinite Popov parameter’, which is the horizontal Popov line in the graphical interpretation of the criterion); we deduce finally

\[
k_\phi + k_H < \frac{1}{T_R} \left(k_p + \frac{1}{T_R L_{\delta_a}}\right).
\]  
\[(39)\]
This corresponds to the case $\omega_e = 0$ in (36); clearly the upper limit of the roll angle overall gain is lower than in the linear case. This is nevertheless a sufficient condition that we may hope to improve using the Yakubovich criterion. This criterion reads
\[
\frac{\tau}{\phi} + \omega^2 \frac{\beta}{\nu} + \omega_e \Re(\tau + i\omega \theta + \beta \omega^2) \left( \frac{1}{i\omega} - L_{\delta_\alpha} k_{\phi} + k_H + i\omega k_P \right) > 0.
\]

The choice $\tau = 0$ is again necessary to simplify $\omega^2$; otherwise, a change of sign would occur. Some simple manipulation that we do not reproduce here will give the condition
\[
\frac{\beta}{\nu} \omega^4 + \left( \frac{\beta}{\nu} \frac{1}{T_R} + \omega_e \theta - \omega_e L_{\delta_\alpha} k_P \beta \right) \omega^2 + \frac{\omega_e \theta}{T_R} - \omega_e L_{\delta_\alpha} \frac{k_{\phi} + k_H}{T_R} - \omega_e L_{\delta_\alpha} \theta (k_{\phi} + k_H - k_P/T_R) > 0.
\]

A necessary condition for this inequality is obtained for $\omega = 0$,
\[
k_{\phi} + k_H < \frac{\theta}{\theta + \beta/T_R} \cdot \frac{1}{T_R} \left( k_P + \frac{1}{T_R L_{\delta_\alpha}} \right).
\]

that gives the worse upper limit unless $\beta = 0$. The Yakubovich criterion does not improve the estimate given by the Popov criterion. However, this should not be surprising, because the estimate is not of the absolute stability sector but of another system parameter.

We shall consider another application from aircraft dynamics—the short period dynamics in the longitudinal modes, with FCS acting as SAS and compliant pilot. The equations are as follows, assuming horizontal flight:
\[
\ddot{\alpha} - M_q \dot{\alpha} - M_\alpha - M_\delta = 0,
\]
\[
\dot{\delta} = \omega_e \psi \left( -(k_\alpha + k_H) \alpha - k_q \dot{\alpha} - \delta \right),
\]
with the actuator being a rate limiter with the same parameters as in the previous case. Here, $\alpha$ is the pitch angle of the aircraft; the parameters are positive, except for $M_q < 0$. This will give an unstable short-period mode of the bare airframe.

In the linear case, when $\psi(v)$ is unsaturated with the slope $V_L/\varepsilon_L$, the characteristic equation of the linear system will be
\[
\sigma^3 + (\omega'_e - M_q) \sigma^2 + (\omega'_e A_q - M_\alpha) \sigma + \omega'_e A_\alpha = 0,
\]
where we denoted $\omega'_e = \omega_e V_L/\varepsilon_L$ as in the previous case and
\[
A_q = M_\delta k_q - M_q > 0, \quad A_\alpha = M_\delta (k_\alpha + k_H) - M_\alpha.
\]

Because of the aircraft inherent stability condition (which is a FCS [SAS] design condition), we shall have $M_\delta k_q - M_q > 0$; therefore, $A_\alpha > 0$. As in the previous application, we will obtain a stability limitation on the global attack angle gain as follows:
\[
k_\alpha + k_H < (\omega'_e - M_q) (k_q - M_q/M_\delta) + \frac{M_q M_\alpha}{\omega'_e M_\delta}.
\]

Consider now the rate limiter being active. The equations (41) display an absolute stability structure as follows:
\[
\ddot{\alpha} - M_q \dot{\alpha} - M_\alpha - M_\delta = 0, \quad \dot{\delta} = -\omega_e \mu(t),
\]
\[
\dot{v} = -(k_\alpha + k_H) \alpha - k_q \dot{\alpha} - \delta, \quad \mu = -\psi(v).
\]

We deduce the transfer function of the linear part
\[
\chi(\sigma) = \frac{\tilde{v}(\sigma)}{\tilde{\mu}(\sigma)} = \frac{\omega_e}{\sigma} \left( 1 + M_\delta \cdot \frac{k_\alpha + k_H + k_q \sigma}{\sigma^2 - M_q \sigma - M_\alpha} \right).
\]
which is very much alike to (38), but, because $M_\alpha > 0$, the system is in an unstable case. In order to apply the standard Popov (and Yakubovich) criterion, we perform the 'sector rotation'. Assume for a while that $\psi(v) = \gamma v$, $\gamma > 0$; the characteristic equation will be

$$\sigma^3 + (\omega_e \gamma - M_q)\sigma^2 + (\omega_e \gamma A_q - M_\alpha)\sigma + \omega_e \gamma A_\alpha = 0.$$  \hspace{1cm} (47)

Denoting $\omega_e \gamma = \xi$, we have $\xi - M_q > 0$ because $M_q < 0$; also, $A_\alpha > 0$. Hence, the Routh Hurwitz inequality will give $\xi > \xi_+$, where $\xi_+$ is the positive root of the equation

$$A_q \xi^2 - (M_\alpha + A_q M_q + A_\alpha)\xi + M_\alpha M_q = 0.$$  

A compatibility condition would be $\xi_+ < \omega'_+ = \omega_e V_L/\varepsilon_L$; this is true by using (44) and continuity arguments. We rewrite (41) as follows:

$$\ddot{\alpha} - M_q \dot{\alpha} - M_\alpha - M_\xi = 0,$$

$$\dot{\xi} + \xi_+((k_\alpha + k_H)\alpha + k_\beta \dot{\alpha} + \delta) = \xi_+((k_\alpha + k_H)\alpha + k_\beta \dot{\alpha} + \delta) + \omega_e \psi(- (k_\alpha + k_H)\alpha - k_\beta \dot{\alpha} - \delta).$$  \hspace{1cm} (48)

We may define $\tilde{\psi}(v) = -\gamma_+ v + \psi(v)$, $\omega_e \gamma_+ = \xi_+$; hence,

$$-\gamma_+ < \tilde{\psi}(v)/v < V_L/\varepsilon_L - \gamma_+, \quad -\gamma_+ < \tilde{\psi}'(v) < V_L/\varepsilon_L - \gamma_+,$$  \hspace{1cm} (49)

and the linear part will have now the transfer function

$$\tilde{\chi}(\sigma) = \omega_e \frac{\sigma^2 + A_q \sigma + A_\alpha}{(\sigma + p_0)(\sigma^2 + \omega_0^2)},$$  \hspace{1cm} (50)

where $p_0 = \xi_+ - M_q > 0$, $\omega_0^2 = A_q \xi_+ - M_\alpha > 0$. The transformed system is thus in the critical case of a pair of purely imaginary roots. The Popov criterion is, as usual,

$$\frac{\tau}{\phi} + \Re e(\tau + i \omega \theta) \tilde{\chi}(i \omega) \geq 0.$$

We took $\phi = 0$, because this is the case for the transformed system (48); within the subsector ($-\gamma_+, 0$), the system is unstable even for linear functions (see previous discussions). Using (50), we find

$$\frac{\tau}{\phi \omega_e} + \frac{(\tau p_0 + \omega^2 \theta)(A_\alpha - \omega^2) + \omega^2 A_q (\tau - \theta p_0)}{(p_0^2 + \omega^2)(\omega_0^2 - \omega^2)} \geq 0,$$

and $\theta/\tau$ must be taken in order to simplify the factor $\omega_0^2 - \omega^2$. A straightforward computation will give

$$\hat{\theta}/\tau = \frac{A_\alpha p_0 + (A_q - p_0)\omega_0^2}{\omega_0^2 (A_q p_0 + \omega_0^2 - A_\alpha)} > 0,$$

and from here,

$$\frac{(\hat{\theta}/\tau)\omega^2 + A_q p_0/\omega_0^2}{p_0^2 + \omega^2} > 0;$$

hence, the frequency domain condition holds for the same parameters as those ensuring linear stability. This is not so obvious, but it follows from the fact that $\hat{\theta}/\tau$, as defined previously, is strictly positive provided (44) holds. This shows that, in this case, the Yakubovich criterion cannot give any improvement.
3.3. An application with two nonlinear elements

We shall consider now the roll subsidence system with two control signals—the elevon and the canard. The system has two identical actuators, and FCSs and the compliant pilot controls both channels. Therefore, the equations of the dynamics are

\[
\ddot{\phi} + \frac{1}{T_R} \dot{\phi} = L_e \delta_e + L_c \delta_c,
\]

\[
\dot{\delta}_e = \omega_e \psi(-(k_\phi + k_H) \dot{\phi} - k_p \ddot{\phi} - \delta_e).
\]

\[
\dot{\delta}_c = \omega_e \psi(-(k_\phi + k_H) \dot{\phi} - k_p \ddot{\phi} - \delta_c).
\]

The saturation level of \(\psi\) is again \(V_L\), whereas the saturation level of the input signal is \(\varepsilon_L\). The characteristic equation of the linearized system is

\[
(L_e + L_c) \omega_e'(k_\phi + k_H + k_p \sigma) = 0.
\]

A simple comparison to (35) will give the Routh Hurwitz inequality (36) with \(L_\delta\) replaced by \(L_e + L_c\).

In the nonlinear case, we shall have the multivariable absolute stability structure

\[
\ddot{\phi} + \frac{1}{T_R} \dot{\phi} = L_e \delta_e + L_c \delta_c,
\]

\[
\dot{\delta}_e = -\omega_e \mu_e(t) , \; \dot{\delta}_c = -\omega_e \mu_c(t),
\]

\[
\nu_e = -(k_\phi + k_H) \dot{\phi} - k_p \ddot{\phi} - \delta_e , \; \nu_c = -(k_\phi + k_H) \dot{\phi} - k_p \ddot{\phi} - \delta_c ,
\]

\[
\mu_e = -\psi(\nu_e) , \; \mu_c = -\psi(\nu_c).
\]

The matrix transfer function of the linear block is

\[
H(s) = \frac{\omega_e}{\sigma} \left[ I_2 + \frac{k_\phi + k_H + k_p \sigma}{\sigma(\sigma + 1/T_R)} \begin{pmatrix} L_e & L_c \\
L_e & L_c \end{pmatrix} \right].
\]

We shall apply first the Popov criterion in dimension 2,

\[
\begin{pmatrix} \tau_e/\dot{\phi}_e & 0 \\
0 & \tau_e/\dot{\phi}_e \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \tau_e + i \omega \theta_e & 0 \\
0 & \tau_e + i \omega \theta_e \end{pmatrix} H(i \omega) + \frac{1}{2} H^*(i \omega) \begin{pmatrix} \tau_e - i \omega \theta_e & 0 \\
0 & \tau_e - i \omega \theta_e \end{pmatrix} > 0,
\]

and make use of the Sylvester conditions. The first one will give

\[
\frac{\tau_e}{\omega_e \dot{\phi}_e} + \theta_e - \frac{L_e}{\omega^2(\omega^2 + 1/T_R^2)} (\tau_e/T_R + \omega^2 \theta_e)(k_\phi + k_H + (\tau_e - \theta_e/T_R) k_p \omega^2) > 0,
\]

and the choice \(\tau_e = 0\), that is, the ‘infinite Popov parameter’ of the critical case with double zero root is obvious. Therefore, we obtain, after some simple manipulation,

\[
k_\phi + k_H < \frac{1}{T_R} \left( k_p + \frac{1}{T_R L_e} \right),
\]

which is of the same type as (39). Considering then the other ‘corner condition’, which is necessary for the fulfilment of the main (determinant) Sylvester condition, we obtain, in a similar way,

\[
k_\phi + k_H < \frac{1}{T_R} \left( k_p + \frac{1}{T_R L_c} \right)
\]

with the choice \(\tau_e = 0\). With these choices for \(\tau_e = \tau_c = 0\), we may write the determinant conditions. Provided the additional choice \(\theta_e L_e = \theta_c L_c\) is made, we obtain

\[
\theta_e \dot{\theta}_e \left[ 1 + (k_p/T_R - k_\phi - k_H) \frac{L_e}{\omega^2 + 1/T_R^2} \right] \left[ 1 + (k_p/T_R - k_\phi - k_H) \frac{L_c}{\omega^2 + 1/T_R^2} \right] > (k_p/T_R - k_\phi - k_H)^2.
\]
From here, we obviously obtain

$$1 + \left(\frac{k_p}{T_R} - k_p - k_H\right) \frac{L_e + L_c}{\omega^2 + 1/T_R^2} > 0,$$

which gives

$$k_p + k_H < \frac{1}{T_R} \left(\frac{k_p}{T_R} + \frac{1}{T_R(L_e + L_c)}\right). \quad (57)$$

If (57) holds, then (55) and (56) are automatically satisfied. This condition is much alike (39) with

$$L_e = L_e + L_c.$$ This shows that the use of two control signals does not improve neither linear nor absolute stability. The application of the Yakubovich-type criterion does not offer any improvement (see (40)).

4. INTEGRAL QUADRATIC CONSTRAINTS AND SOME PROOF BACKGROUND

We shall start from the sector and slope restrictions (24). Let $\sigma_j(t)$ be some signal (a function of the real variable $t$ — the ‘time’ — mathematically speaking) and define the following signals

$$\xi_j(t) = -\varphi_j(\sigma_j(t)), \quad \mu_j(t) = \dot{\xi}_j(t) \quad (58)$$

(assuming, of course, differentiability conditions as fulfilled). Following, for example, [10, 15, 19, 20], we may write the following inequalities associated with one nonlinear function:

$$(\xi_j + \varphi^j(\sigma_j))(\xi_j + \bar{\varphi}^j(\sigma_j)) = (\varphi^j(\sigma_j - \varphi_j(\sigma_j)) \bar{\varphi}^j - \varphi_j(\sigma_j))^2 \leq 0, \quad (59)$$

$$\int_0^T (\xi_j(t) + \varphi^j(\sigma_j(t))) \dot{\sigma}_j(t) dt = \int_0^T (\varphi^j(\sigma_j(t) - \varphi_j(\sigma_j(t))) \dot{\sigma}_j(t) dt = \frac{1}{2} \varphi^j \sigma^2_j(T)$$

$$- \int_0^{\sigma_j(0)} \varphi_j(\lambda) d\lambda - \frac{1}{2} \bar{\varphi}^j \sigma^2_j(0) + \int_0^{\sigma_j(T)} \varphi_j(\lambda) d\lambda \leq \int_0^{\sigma_j(0)} \varphi_j(\lambda) d\lambda - \frac{1}{2} \varphi^j \sigma^2_j(0), \quad (60)$$

$$- \int_0^T (\xi_j(t) + \bar{\varphi}^j(\sigma_j(t))) \dot{\sigma}_j(t) dt = \frac{1}{2} \bar{\varphi}^j \sigma^2_j(T) + \int_0^{\sigma_j(0)} \varphi_j(\lambda) d\lambda$$

$$+ \frac{1}{2} \bar{\varphi}^j \sigma^2_j(0) - \int_0^{\sigma_j(0)} \varphi_j(\lambda) d\lambda \leq \frac{1}{2} \varphi^j \sigma^2_j(0) - \int_0^{\sigma_j(0)} \varphi_j(\lambda) d\lambda, \quad (61)$$

$$(\mu_j(t) + \bar{\varphi}^j \dot{\sigma}_j(t))(\mu_j(t) + \bar{\varphi}^j \dot{\sigma}_j(t)) = (\varphi^j - \varphi_j(\sigma_j(t))) \bar{\varphi}^j - \varphi_j(\sigma_j(t))^2 \sigma_j(t)^2 \leq 0. \quad (62)$$

It may happen that one of the sector and/or slope restrictions is infinite. For instance, let $\bar{\varphi}^j \rightarrow \infty$, which implies $\bar{\varphi}^j \rightarrow \infty$. In this case, some of the aforementioned restrictions will be replaced as follows:

$$(\xi_j + \varphi^j(\sigma_j)) \sigma_j = (\varphi^j(\sigma_j - \varphi_j(\sigma_j)) \sigma_j \leq 0, \quad (63)$$

$$- \int_0^T \sigma_j(t) \dot{\sigma}_j(t) dt = \frac{1}{2} \sigma_j(0)^2 - \frac{1}{2} \sigma_j(T)^2 \leq \frac{1}{2} \sigma_j(0)^2, \quad (64)$$

$$(\mu_j(t) + \bar{\varphi}^j \dot{\sigma}_j(t)) \dot{\sigma}_j(t) = (\bar{\varphi}^j - \varphi_j(\sigma_j(t))) \dot{\sigma}_j(t)^2 \leq 0. \quad (65)$$

Obviously, if only $\bar{\varphi} \rightarrow \infty$ but $\varphi_j$ stands finite, then only (62) is modified in (65). Because the case of the lower sector and slope restrictions approaching $-\infty$ is similar, we skip it.
We have already addressed twice this question of the semi-infinite sectors. Here, we want only to mention that semi-infinite sectors and slope bounds lead to modifications of the quadratic constraints and, consequently, of the quadratic integral index (see next). It will also appear in the sequel that modification of the quadratic constraints has, as a consequence, a corresponding modification of the frequency domain inequality. Especially, the behavior for \( \omega \to \infty \) is affected, and additional assumptions are required; this aspect has been discussed previously in Section 3.1.

It is also interesting to point out that all aforementioned inequalities deal with quadratic forms with respect to their variables. Some of them, as (59), (62), (63), (65), are valid for all real \( t \), that is, are ‘static’ or ‘local’. The other ones are ‘dynamic’ or ‘integral’. The use of such constraints goes back to [13,15], being largely generalized in [21], where also frequency domain IQC are considered (see [52] for an account on this subject).

Another mathematical object related to absolute stability proofs is the so-called Popov system. The use of this system underlines the absolute stability since the very first paper of V. M. Popov on the frequency domain approach [11]. It always consists in a controlled linear system and an integral index defined along the solutions of the linear system. The structure of this index appeared as arbitrary (\( a \) priori integral index—to cite [44]) up to the moment when it became clear that its structure is as in \([19,20,38]\). We shall have

\[
\dot{x} = Ax + \sum_{i=1}^{m} b_i \xi_j , \quad \dot{\xi}_j = \mu_j(t) , \quad j = 1, m ,
\]

or, if the vector notation is used,

\[
\dot{x} = Ax + Bz , \quad \dot{z} = u(t).
\]

The connection of this system, that is, of its solutions to the basic system (23) is rather simple. Let \( x(t) \) be some solution of (23). Defining the initial conditions for (66) and the control functions \( \mu_j(t) \) by

\[
x_0 = x(0) , \quad \dot{x}_0 = -\varphi_j(c^*_j x(0)) , \quad \mu_j(t) = -\frac{d}{dt} \varphi_j(c^*_j x(t)) , \quad j = 1, m.
\]

we may construct the corresponding solution, that is, state trajectory \((\tilde{x}(t), \xi_j(t))\) of (66). It follows in a straightforward way that \( \tilde{x}(t) \equiv x(t) \), the solution of (23). For this reason, we shall use for the solutions of (66) the same notation \( x(t) \) even if this will look like an abuse of notation due to some definite steps of the presentation.

The integral index is now associated by using the constraints (59)–(62) but with a slight modification in order to incorporate the cases when some \( \overline{\nu}^j \) and/or \( \overline{\varphi}^j \) approach \( +\infty \). As already mentioned, we shall skip the counterpart of this case when some \( \nu^j \) and/or \( \varphi^j \) approach \(-\infty\).

\[
\eta_j(0, T) = \tau_j \int_0^T (\xi_j(t) + \varphi^j(c^*_j x(t))) \dot{\xi}_j(t) / \overline{\varphi}^j + c^*_j x(t)) dt \\
+ \theta_j \int_0^T \xi_j(t) + \varphi^j(c^*_j x(t)) c^*_j x(t) dt - \theta_j \int_0^T (\xi_j(t) / \overline{\varphi}^j + c^*_j x(t)) \dot{x}(t) dt \\
+ \beta_j \int_0^T (\mu_j(t) + \nu_j c^*_j x(t)) \dot{x}(t) / \overline{\varphi}^j + c^*_j x(t) dt \\
= \tau_j \int_0^T (\xi_j(t) + \varphi^j(c^*_j x(t))) \dot{\xi}_j(t) / \overline{\varphi}^j + c^*_j x(t)) dt \\
+ \frac{1}{2} (\theta_j \varphi^j - \theta_j \nu_j)(c^*_j x(t))^2 \int_0^T \xi_j(t) c^*_j(Ax(t) + \sum_{i=1}^{m} b_i \xi_i(t)) dt \\
+ \beta_j \int_0^T (\mu_j(t) + \nu_j c^*_j(Ax(t) + \sum_{i=1}^{m} b_i \xi_i(t))) \dot{x}(t) / \overline{\varphi}^j + c^*_j(Ax(t) + \sum_{i=1}^{m} b_i \xi_i(t)) dt.
\]
The associated integral index is defined as

$$\eta(0, T) = \sum_{1}^{m} \eta_j(0, T). \quad (70)$$

Remark that all $\eta_j(0, T)$ are quadratic forms with respect to their arguments. Considering the vector arguments of the quadratic forms, that is, $u, x,$ and $z$, the overall integral index may be written as

$$\eta(0, T) = x^*(t)Jx(t)\bigg|_0^T + \int_{0}^{T} \mathcal{F}(u(t), x(t), z(t))dt, \quad (71)$$

where we denoted

$$\mathcal{F}(u, x, z) = u^*Ku + u^*L_1^*x + u^*L_2^*z + x^*L_1u + z^*L_2u + x^*M_{11}x + x^*M_{12}z + z^*M_{12}x + z^*M_{22}z \quad (72)$$

and

$$J = \frac{1}{2}C(D_1E - D_2^{e})C^*; \quad K = D_3G^{-1}, \quad L_1^* = \frac{1}{2}D_3(I + GG^{-1})C^*A, \quad \text{(73)}$$

$$L_2^* = \frac{1}{2}D_3(I + GG^{-1})C^*B, \quad M_{11} = CE_{11}C^* + A^*CGD_3C^*A,$$

$$M_{12} = \frac{1}{2}D_1(I + EF^{-1})C^* + \frac{1}{2}(D_2^{e} - D_2^{e}F^{-1})C^*A + A^*CGD_3C^*B,$$

$$M_{22} = D_1F^{-1} + \frac{1}{2}((D_2^{e} - D_2^{e}F^{-1})C^*B + B^*C(D_2^{e} - D_2^{e}F^{-1})) + B^*CGD_3C^*B.$$ 

Obviously, system (66) has an augmented dynamics with respect to the initial system (23). We may thus introduce the augmented state vector $x_a$ and the Popov system with augmented dynamics

$$\dot{x}_a = A_a x_a + B_a u(t), \quad \eta_a(0, T) = x_a(t)^*Jax_a(t)\bigg|_0^T + \int_{0}^{T} \mathcal{F}_a(u(t), x_a(t))dt. \quad (74)$$

From now on, we follow the line of [10, Chapter 2]. The characteristic matrix function of the Popov system (74) is, by definition,

$$\mathcal{H}(\lambda, \sigma) = K + L_a^*b(\sigma I - A_a)^{-1}B_a + B_a^*(\lambda I - A_a)^{-1}L_a$$

$$+ B_a^*(\lambda I - A_a)^{-1}(M_a + (\lambda + \sigma)J_a)(\sigma I - A_a)^{-1}B_a. \quad (75)$$

If we take into account the previous notations, then

$$\mathcal{H}(\lambda, \sigma) = \frac{1}{\lambda\sigma} \left\{D_1F^{-1} + \frac{1}{2}D_1(I + EF^{-1})H(\sigma) + \frac{1}{2}H^*(\lambda)(I + EF^{-1})D_1 + H^*(\lambda)D_1EFH(\sigma) + \frac{1}{2}\lambda H^*(\bar{\lambda})(D_2^{e} - D_2^{e}F^{-1}) + \frac{1}{2}(D_2^{e} - D_2^{e}F^{-1})\sigma H(\sigma) + \frac{1}{2}(\lambda + \sigma)H^*(\bar{\lambda})(D_2^{e}F - D_2)H(\sigma) + \lambda\sigma \left[ D_3G^{-1} + \frac{1}{2}D_3(I + GG^{-1})H(\sigma) - \frac{1}{2}H^*(\bar{\lambda})I + GG^{-1})D_3 + H^*(\bar{\lambda})D_3G^2H(\sigma) \right] \right\}. \quad (76)$$

Here, $\lambda , \quad \sigma \in \mathbb{C}$, and $H(\sigma) = C^*(\sigma I - A)^{-1}B$ is the matrix transfer function defined by (26)–(27). Because, in general, we considered complex coefficients, we have $H^*(\lambda) = (H(\bar{\lambda}))^T,$ the $T$ denoting ‘transpose’.

We shall consider now a modified system arising from ‘sector rotation’. Starting from the system (23), with the nonlinearities satisfying (24), we introduce

$$\psi_j(v) = -\bar{\varphi}_j v + \varphi_j(v) \quad (77)$$
for some \( \tilde{\varphi}_j \in [\varphi^l, \varphi^u] \subseteq [\bar{\varphi}^l, \bar{\varphi}^u] \), which, in particular, may be those of Definition 1. It follows from (24) that
\[
\varphi^l - \tilde{\varphi}_j \leq \psi_j(v)/\nu \leq \varphi^l - \tilde{\varphi}_j, \quad \nu^l - \tilde{\varphi}_j \leq \psi_j'(v) \leq \nu^l - \tilde{\varphi}_j, \quad j = 1, m.
\] (78)

We may now define the associated Popov system
\[
\tilde{x} = A_F \tilde{x} + B \tilde{z}, \quad \tilde{z} = u(t),
\] (79)
where we denoted
\[
\mathcal{F}(\tilde{u}, \tilde{x}, \tilde{z}) = \tilde{u}^* \tilde{K} \tilde{u} + \tilde{u}^* \tilde{L}_1 \tilde{x} + \tilde{z}^* \tilde{L}_2 \tilde{x} + \tilde{z}^* \tilde{M}_1 \tilde{x} + \tilde{x}^* \tilde{M}_2 \tilde{z} + \tilde{z}^* \tilde{M}_3 \tilde{z},
\] (80)

and \( \bar{\varphi}_j \) is the diagonal matrix with \( \tilde{\varphi}_j \) as main diagonal elements. The two systems are equivalent, in the sense that they belong to the same class as defined in [10, Chapter 2, Definition 1]. Accordingly, the relation between the two characteristic matrix functions is
\[
\mathcal{H}(\lambda, \sigma) = (I_m + H^*(\tilde{\lambda})\tilde{F}). \mathcal{H}(\lambda, \sigma)(I_m + \tilde{F} H(\sigma))
\] (83)
(Remember that \( \tilde{F} \) is symmetric, diagonal with real entries).

Following [10, Chapter 2], we introduce the frequency domain matrix characteristic function
\[
\mathcal{H}(-i\omega, i\omega) = \frac{1}{\omega^2} \left[ D_1 \tilde{F}^{-1} + \Re[D_1 (I + \tilde{F})^{-1} + i\omega(D_1'D_2'D_2'\tilde{F}^{-1})]H(i\omega) + H^*(-i\omega)D_1 F H(i\omega) + \omega^2[D_3 \tilde{G}^{-1} + \Re[D_3 (I + \tilde{G})^{-1}]H(i\omega)
\right]
\] (84)

Remark that
\[
\omega^2 \mathcal{H}(-i\omega, i\omega) \equiv T(i\omega),
\] (85)
where \( T(i\omega) \) is that of the frequency domain inequality in Section 3, Theorem 1 (the main result), with \( D_2 = D_2' - D_2' \tilde{F}^{-1} \). Now, if the frequency domain condition holds for \( D_2 \geq 0 \), we may take \( D_2' = D_2, D_2'' = 0 \); if it holds for \( D_2 < 0 \), we may take \( D_2' = 0, D_2'' = D_2 \tilde{F} \). Because this choice is, in fact, componentwise, the aforementioned rules may be combined for a \( D_2 \) that has not a definite sign. From (83), we deduce
\[
\mathcal{H}(-i\omega, i\omega) = (I_m + H^*(-i\omega)\tilde{F}). \mathcal{H}(-i\omega, i\omega)(I_m + \tilde{F} H(i\omega));
\] (86)
hence, if $\mathcal{H}(-i\omega, i\omega)$ has definite sign (e.g., nonnegative) for all real $\omega$, the same holds for the frequency domain characteristic $\hat{\mathcal{H}}(-i\omega, i\omega)$ of an equivalent function.

In order to make the paper self-contained, we reproduce here, from [44], a Liapunov-like lemma.

**Lemma 1**
Consider the system of ordinary differential equations in the general form
\[ \dot{x} = f(x), \quad \text{dim} x = \text{dim} f = n. \] (87)

Any constant vector $c$ satisfying $f(c) = 0$ is called a stationary vector; the set of all stationary vectors is called stationary set. Assume there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ with the following properties: (i) $V(x(t))$ is nonincreasing with respect to $t$ along any solution of (87); (ii) if for some bounded for $-\infty < t < \infty$ $x(t)$, $V(x(t)) \equiv \text{const}$, then this solution is a stationary vector; and (iii) $\lim_{|x| \to \infty} V(x) = \infty$. Then, system (87) has global asymptotics, that is, any of its solutions approaches for $t \to \infty$, the stationary set.

5. PROOF OF THEOREM 1

We shall follow here the approach of [38], at its turn based on [19, 20]. This approach will be combined with the results of [10], namely Theorem 1 of Chapter 5 (generalized to the multivariable case). Some general results of Chapter 2 of [10], already presented in Section 4 of this paper, will be also applied.

We start from the basic system (23), where the nonlinear functions satisfy the sector and slope restrictions (24). Following Section 4, we associate the Popov system (66)—or (67)—with the integral index (71) whose quadratic form is defined by (72) and its matrices by (73). The characteristic matrix function is given by (76). The nonnegative diagonal matrices of free parameters $D_1, D_2, D_2', D_3'$ are chosen starting from the parameters ensuring fulfilment of the frequency domain inequality (27) of Theorem 1. More precisely, $D_1$ and $D_3$ are exactly those of (27), whereas $D_2$ and $D_2'$ are chosen following the procedure described in Section 4. Accordingly, we shall have the sign of $\mathcal{H}(-i\omega, i\omega)$ coinciding with the sign of $T(i\omega)$ in Theorem 1. Moreover, because $(A, B)$ is a controllable pair, the pair $\left( \begin{array}{cc} A & B \\ 0 & 0 \end{array} \right), \left( \begin{array}{c} 0 \\ I_m \end{array} \right)$ is also controllable. Indeed,

\[
\text{rank} \begin{pmatrix} 0 & B & AB & \ldots & A^{n+m-2}B \\ I_m & 0 & 0 & \ldots & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & B & AB & \ldots & A^{n-1}B \\ I_m & 0 & 0 & \ldots & 0 \end{pmatrix} = m + \text{rank} \begin{pmatrix} 0 & B & AB & \ldots & A^{n-1}B \end{pmatrix} = m + n,
\]

where the first equality is due to the Cayley Hamilton theorem.

We are now in position to apply Positiveness Theorem for multi-input systems [10, Chapter 3, Theorem 9.1] to the Popov system (66)/(67) and (71)-(73). This theorem incorporates the Yakubovich–Kalman–Popov lemma, and we shall apply it in the two cases of Theorem 1—the noncritical and the critical.

First, let $A$ have its eigenvalues outside the imaginary axis $i\mathbb{R}$ and the frequency domain inequality (27) be strict for $\omega \in \mathbb{R}$, that is, including $\omega \to \infty$. In this case, there exists some $\varepsilon > 0$ such that the modified frequency domain condition holds
\[ T(i\omega) - \varepsilon B^*(-i\omega I - A^*)^{-1}(-i\omega I - A)^{-1}B \geq 0. \] (88)

This suggests introduction of the modified integral index by taking
\[ \mathcal{F}_\varepsilon(u, x, z) = \mathcal{F}(u, x, z) - \varepsilon x^*(t)x(t), \] (89)
that is, by modifying the matrix $M_{11}$ of (72) and (73) as $M_{11}^\varepsilon = M_{11} - \varepsilon I_n$. Positiveness Theorem, cited earlier, applied to this modified system, gives existence of the matrices $V^\varepsilon, W_1^\varepsilon, W_2^\varepsilon,$
With $N_{11}^\varepsilon = (N_{11}^\varepsilon)^*$, $N_{12}^\varepsilon$, $N_{22}^\varepsilon = (N_{22}^\varepsilon)^*$ of appropriate dimensions such that $\eta^\varepsilon(0,T)$ defined with $\mathcal{F}_x(u,x,z)$, via (71)–(73) but with $M_{11}^\varepsilon$ previously, considered along the solutions of (67) reads

$$
\eta^\varepsilon(0,T) = \frac{1}{2} \sum_{j=1}^{m} \left[ \theta_j^\varepsilon - \theta_j^\varepsilon \right] (c_j^\varepsilon x(t))^2 |^T_0 \\
- (x^*(t) N_{11}^\varepsilon x(t) + x^*(t) N_{12}^\varepsilon z(t) + z^*(t) (N_{12}^\varepsilon)^* x(t) + z^*(t) N_{22}^\varepsilon z(t)) |^T_0 \\
+ \int_0^T | V^\varepsilon u(t) + (W_1^\varepsilon)^* x(t) + (W_2^\varepsilon)^* z(t) |^2 dt.
$$

Now, let $x(t)$ be a solution of (23), corresponding to some initial condition and to some set of nonlinear functions subject to (24). We take next

$$
\mu_j(t) = -\frac{d}{dt} \varphi_j(c_j^\varepsilon x(t)) , \quad x_0 = x(0) , \quad \xi_0^j = -\varphi_j(c_j^\varepsilon x(0)).
$$

If $(\bar{x}(t), \xi_j(t), j = 1, m)$ is the corresponding solution of (66), it follows via a straightforward proof that $\bar{x}(t) \equiv x(t)$; therefore, we drop the bar without any abuse of notation. For this solution of (66), we shall have $\eta^\varepsilon(0,T)$ as

$$
\eta^\varepsilon(0,T) = \sum_{j=1}^{m} \eta_j(0,T) - \varepsilon \int_0^T x^*(t) x(t) dt,
$$

where

$$
\eta_j(0,T) = \tau_j \int_0^T (\varphi_j^\varepsilon v_j(t) - \varphi_j(v_j(t)))(v_j(t) - \varphi_j(v_j(t))/\varphi_j^\varepsilon) dt + \theta_j^\varepsilon \int_{v_j(0)}^{v_j(T)} (\varphi_j^\varepsilon - \varphi_j(\lambda)) d\lambda \\
- \theta_j^\varepsilon \int_{v_j(0)}^{v_j(T)} (\lambda - \varphi_j(\lambda)/\varphi_j^\varepsilon) d\lambda + \varepsilon_j \int_0^T (\dot{v}_j^\varepsilon - \varphi_j(v_j(t)))(1 - \varphi_j(v_j(t))/\varphi_j^\varepsilon)(\dot{v}_j(t))^2 dt.
$$

We want to equate the two expressions of $\eta^\varepsilon(0,T)$; this suggests introduction of the following state function:

$$
\gamma^\varepsilon(x) = -(x^* N_{11}^\varepsilon x - f(C^* x)^* (N_{12}^\varepsilon)^* x - x^* N_{12}^\varepsilon f(C^* x) + f(C^* x)^* N_{22}^\varepsilon f(C^* x)) \\
+ \sum_{j=1}^{m} (\theta_j^\varepsilon - \theta_j^\varepsilon/\varphi_j^\varepsilon) \int_0^x \varphi_j(\lambda) d\lambda.
$$

Equating the two forms of $\eta^\varepsilon(0,T)$ and using the aforementioned introduced state function, we deduce

$$
\gamma^\varepsilon(x(T)) = \gamma^\varepsilon(x(0)) - \int_0^T | -V^\varepsilon \frac{d}{dt} f(C^* x(t)) + (W_1^\varepsilon)^* x(t) - (W_2^\varepsilon)^* f(C^* x(t)) |^2 dt \\
- \tau_j \int_0^T (\varphi_j^\varepsilon v_j(t) - \varphi_j(v_j(t)))(v_j(t) - \varphi_j(v_j(t))/\varphi_j^\varepsilon) dt \\
- \varepsilon \int_0^T | x(t) |^2 dt.
$$

Taking into account (24), it follows that

$$
\frac{d}{dt} \gamma^\varepsilon(x(t)) \leq -\varepsilon |x(t)|^2,
$$

that is, in the noncritical case, the candidate Liapunov function $\gamma^\varepsilon(x)$ defined by (94) has a negative-definite derivative along the solutions of (23).

Consider now the critical case, when $A$ is allowed to have some eigenvalues on $i \mathbb{R}$: the frequency inequality is nonstrict, whereas the sector and slope restrictions are strict.
The Positiveness Theorem will be applied now for the basic Popov system, which corresponds to \( \varepsilon = 0 \). This will ensure existence of the matrices \( V^o, W^o, W_1^o, N_{11}^o = (N_{11})^o, N_{12}^o, N_{22}^o = (N_{22})^o \). These matrices may be obtained also by letting \( \varepsilon \to 0 \) in \( V^\varepsilon, W_1^\varepsilon \), and so on (continuity reasons). We obtain further \( \mathcal{V}^o(x) \) by letting \( \varepsilon \to 0 \) in (94). Letting \( \varepsilon \to 0 \) in (95), we obtain that the derivative of the candidate Lyapunov function \( \mathcal{V}^o(x) \) along the solutions of (23) is nonpositive definite; making use of the strict sector and slope inequalities, we deduce that this derivative vanishes on the set where

\[
c_j^* x = 0 \quad c_j^* (Ax - \sum_{i=1}^{m} h_i \varphi_i (e_i^* x)) = 0.
\]

Because \( \varphi_j(0) = 0 \), we may use the observability of the pair \((C^*, A)\) to obtain that \{0\} is the only invariant set where the derivative of \( \mathcal{V}^o(x) \) along the solutions of (23) vanishes.

We need now some sign information about the candidate Lyapunov function \( \mathcal{V}^o(x) \). Consider system (23) with linear functions \( \varphi_j(x) = \gamma_j x \) where the sector and slope restrictions hold:

\[
\underline{\gamma}^j \leq \varphi^j \leq \overline{\gamma}^j \leq \overline{\varphi}^j \leq \overline{\underline{\gamma}}^j \quad j = 1, m.
\]

Consider \( F \), the diagonal matrix with the diagonal elements \( \gamma_j \), and the linear system

\[
\dot{x} = (A - BFC^*)x.
\]

By taking \( f(C^* x) = FC^* x \) in \( \mathcal{V}^e(x) \) or \( \mathcal{V}^o(x) \), we associate the quadratic candidate Lyapunov functions for (99)

\[
\mathcal{V}^o_F (x) = -x^* (N_{11}^o - CF(N_{11}^o)^* - N_{12}^o FC^* + CF(N_{22}^o)^* + \frac{1}{2} C(D_2 - D_2^T)^{-1} FC^*) x
\]

and the corresponding one \( \mathcal{V}^e_F (x) \).

Let us first consider \( F = \hat{F} \), where \( \hat{F} \) is defined by the numbers \( \hat{\varphi}^j, j = 1, m \), of Theorem 1: by assumption, (99) is exponentially stable for \( F = \hat{F} \). Consequently, \( \mathcal{V}^o_F (x) \) is positive definite because of (96). In the critical cases, the derivative of \( \mathcal{V}^o_F (x) \) along the solutions of (99) will satisfy, according to (95),

\[
\frac{d}{dt} \mathcal{V}^o_F (x(t)) \leq -\varepsilon_0 |C^* x(t)|^2,
\]

and the observability of \((C^*, A)\) ensures that \( \mathcal{V}^o_F (x) \) is also positive definite.

We prove now that (99) is exponentially stable for all \( F \) satisfying the sector and slope restrictions (98). Let \( \hat{F} = F - \hat{F} \), where \( \hat{F} \) is the diagonal matrix defined previously. We shall thus have, instead of (98), the inequalities

\[
\underline{\gamma}^j - \hat{\varphi}^j \leq \varphi^j - \hat{\varphi}^j \leq \overline{\gamma}^j - \hat{\varphi}^j \leq \overline{\varphi}^j - \hat{\varphi}^j \quad j = 1, m,
\]

and system (99) reads

\[
\dot{x} = (A_{\hat{F}} - B_{\hat{F}} C^*)x,
\]

where \( A_{\hat{F}} = A - B_{\hat{F}} C^* \) is a Hurwitz matrix. We may associate with this linear system, under restrictions (102) (corresponding to the ‘rotated sectors’), the Popov system (80)–(83). This system is equivalent to the initial one defined by (67), (70)–(73), in the sense presented in the previous section. Using (83), we may associate the matrix frequency domain characteristic

\[
\mathcal{F}(-j\omega, j\omega) = \frac{1}{\omega^2} \left\{ D_1 (\hat{F} - \hat{F})^{-1} + \Re (D_1 (I + (E - \hat{F})(\hat{F} - \hat{F})^{-1})
+ j\omega (D_2 - D_2^T (\hat{F} - \hat{F})^{-1}) H_\mathcal{F}(j\omega) + H_\mathcal{F}^*(j\omega) D_1 (E - \hat{F}) H_\mathcal{F}(j\omega)
+ \omega^2 \left[ D_3 (\hat{G} - \hat{F})^{-1} + \Re D_3 (I + (\hat{G} - \hat{F})(\hat{G} - \hat{F})^{-1}) H_\mathcal{F}(j\omega)
+ H_\mathcal{F}^*(-j\omega) D_3 (\hat{G} - \hat{F}) H_\mathcal{F}(j\omega) \right] \right\},
\]

where
\[
H_F(\sigma) = C^*(\sigma I - A_F)^{-1} B = C^*(\sigma I - A + B \hat{F} C^*)^{-1} B = H(\sigma)(I + \tilde{F} Y(\sigma))^{-1}.
\] (105)

We deduce from (86) that \(\mathcal{H}(-\iota \omega, \iota \omega) \geq 0\) implies \(\tilde{\mathcal{H}}(-\iota \omega, \iota \omega) \geq 0\). The same property holds also for \(T(\iota \omega) = \omega^2 \mathcal{H}(-\iota \omega, \iota \omega)\) and \(\tilde{T}(\iota \omega) = \omega^2 \tilde{\mathcal{H}}(-\iota \omega, \iota \omega)\). Moreover, \(\det (I_m + \tilde{F} H(\iota \omega)) \neq 0\) for all real \(\omega\) because \(A_F\) is Hurwitz; therefore, the strict inequality, if valid for the initial system, holds for the transformed also. Also, \(I_m + \tilde{F} H(\sigma)\) is proper, and \(\lim_{\omega \to \infty} (I_m + \tilde{F} H(\sigma)) = I_m\); hence, the frequency domain inequality is preserved too for the transformed system, when \(\omega \to \infty\).

We are now in position to prove the exponential stability of (99) for all \(F\) satisfying the restrictions (98) by proving the exponential stability of (103) for all \(\tilde{F}\) satisfying (102). Were this assertion not true, there would exist a set of numbers \(\hat{\varphi}_j\) satisfying (102) such that system (103) would not be exponentially stable for this particular \(\tilde{F}\). We deduce existence of some \(\omega_0 \in \mathbb{R}\) such that

\[
\det (i \omega_0 I - A_F + B \hat{F} C^*) = \det (i \omega_0 I - A_{\tilde{F}}) \det (I + \tilde{F} H_{F}(i \omega_0)) = 0,
\] (106)

where \(H_{F}(\sigma)\) is that of (105). Because \(A_{\tilde{F}}\) is a Hurwitz matrix, only the second determinant might be 0; hence, there exists a vector \(v_0 \neq 0\) such that \(v_0 = -\tilde{F} H_{F}(i \omega_0)v_0\).

We consider the frequency domain inequality that gives \(T(\iota \omega) \geq 0\) for \(\omega = \omega_0\) and use the condition \(v_0 \tilde{T}(i \omega_0)v_0 \geq 0\) after taking \(H_{F}(i \omega_0)v_0 = -\tilde{F}^{-1}v_0\). Remark that the last equality requires all \(\hat{\varphi}_j \neq 0\). If this is not the case, we consider \(v_0\) with zero components on those positions where \(\hat{\varphi}_j = 0\). Making use of (83), we obtain

\[
v_0^* \tilde{T}(i \omega_0)v_0 = v_0^* \{D_1(F - \tilde{F})^{-1} + \Re(D_1(I + (F - \tilde{F})(\tilde{F} - F)^{-1}) + i \omega_0 D_2)H_F(i \omega_0) + H_F^*(-i \omega_0)D_1(F - \tilde{F})H_F(i \omega_0) + \omega_0^2 D_3(G - \tilde{F})^{-1}\}v_0
\]

\[
+ \Re(D_3(I + (G - \tilde{F})(\tilde{F} - F)^{-1})H_F(i \omega_0) + H_F^*(-i \omega_0)D_3(G - \tilde{F})H_F(i \omega_0))v_0 = v_0^* \{D_1(F - \tilde{F})^{-1} + \Re(D_1(I + (F - \tilde{F})(\tilde{F} - F)^{-1}) + i \omega_0 D_2) \tilde{F}^{-1}
\]

\[
+ \tilde{F}^{-1} D_1(F - \tilde{F}) \tilde{F}^{-1} + \omega_0^2 [D_3(G - \tilde{F})^{-1}\}
\]

\[
- \Re(D_3(I + (G - \tilde{F})(\tilde{F} - F)^{-1}) \tilde{F}^{-1} + \tilde{F}^{-1} D_3(G - \tilde{F}) \tilde{F}^{-1})\}
\]

\[
v_0^2 \sum_j \left[ \tau_j \left( \frac{1}{\varphi_j - \tilde{\varphi}_j} - 1 \right) \left( 1 + \frac{v^j - \tilde{\varphi}_j}{\varphi^j - \tilde{\varphi}_j} \right) + \frac{v^j - \tilde{\varphi}_j}{\varphi^j - \tilde{\varphi}_j} \right] \right]^2.
\]

\[
= \sum_j \left[ \tau_j \left( \frac{1}{\varphi^j - \tilde{\varphi}_j} - 1 \right) \left( 1 + \frac{\varphi^j - \tilde{\varphi}_j}{\varphi^j - \tilde{\varphi}_j} \right) \right] \left( \frac{v^j}{\varphi^j} \right)^2 \geq 0.
\]

Because \(\hat{\varphi}_j \leq \tilde{\varphi}_j \), the first factor multiplying \(\tau_j\) is nonpositive, and because \(\hat{\varphi}_j \geq \tilde{\varphi}_j \), the second one is nonnegative. The same is valid for the factors multiplying \(\beta_j\). Therefore, the aforementioned inequality appears both nonnegative and nonpositive. But if \(T(\iota \omega) > 0\), then \(\tilde{T}(i \omega_0) > 0\) and also \(\tilde{T}(i \omega_0) > 0\), which contradicts the inequalities established previously. If \(T(\iota \omega) \geq 0\) only, then (107) is not strict, but in this case, we assumed strict sector and slope restrictions; hence, the sum of products in (107) is negative, again a contradiction.

A special case occurs when \(\omega_0 = 0\). We have to discuss here two cases. If \(T(0) > 0\), then

\[
\tilde{T}(0) = (I + H^*(0)\tilde{F})T(0)(I + \tilde{F} H(0)) > 0
\]
and the proof goes as previously. If $T(0) = 0$, there are again two cases. If $\sigma = 0$ is not an eigenvalue of $A$, then it is not a pole of $H(\sigma)$, $H(0)$ makes sense, and inequality (28) holds. We have $v_0 = -\tilde{F} H(0) v_0$ for some vector $v_0 \neq 0$. Defining $w_0 = (I + \tilde{F} H(0))^{-1} v_0$, we obtain $w_0 = -(\tilde{F} + \tilde{F}) H(0) w_0$ with the entries of $w_0$ corresponding to $\tilde{\varphi}_j = \tilde{\varphi}_j = 0$, taking also 0. Consider the matrix of (28) and write down the corresponding quadratic form with $w_0$ constructed previously. It follows that

$$\frac{1}{2} w_0^*(\tilde{F} - H(0)) I + (\tilde{F} - H(0)) w_0 + \frac{1}{2} w_0^* (I + H(0)) (\tilde{F} - H(0)) w_0$$

$$= \sum_j \left( \frac{1}{\varphi_j} - \frac{1}{\tilde{\varphi}_j} \right) \left( 1 - \frac{\varphi_j}{\tilde{\varphi}_j} \right) (\sigma_j)^2 \leq 0.$$ 

This contradicts (28). Now, let $\sigma = 0$ be an eigenvalue of $A$, of some multiplicity $r$.\footnote{Because of the assumptions of controllability and observability, we may write $H(\sigma) = \sigma^{-r} W(\sigma)$ with $W(0)$ finite. Consider the matrix}

$$U(\lambda, \sigma) = \frac{1}{2} [(\tilde{F} - \tilde{F})^{-1} + H^* \tilde{F}(\lambda)(I + (\tilde{F} - \tilde{F}) H(\sigma)) + (I + H^* \tilde{F}(\lambda)(\tilde{F} - \tilde{F}))((\tilde{F} - \tilde{F})^{-1} + H^* \tilde{F}(\sigma))]$$

with $\lambda, \sigma \in \mathbb{C}$. A straightforward manipulation will give

$$U(\lambda, \sigma) = \frac{1}{2} (\lambda^r I + W^*(\lambda) \tilde{F})^{-1} [(\lambda^r I + W^*(\lambda) \tilde{F})(\tilde{F} - \tilde{F})^{-1}(\sigma^r I + F W(\sigma)) + (\sigma^r I + F W(\sigma)) (\tilde{F} - \tilde{F})^{-1} (\sigma^r I + F W(\sigma))^{-1}].$$

Because $\tilde{F}$ ensures $A - B \tilde{F}^* C$ to be a Hurwitz matrix, its elements may ‘accept’ a sufficiently small modification that does not alter the Hurwitz character of $A$ but makes $\tilde{F}$ invertible (with all diagonal elements nonzero). We already know from previous development that $I + \tilde{F} H(\sigma)$ is invertible for all $\sigma \in \mathbb{C}$. We may thus write

$$\lim_{\lambda, \sigma \to 0} U(\lambda, \sigma) = (W^*(0) \tilde{F})^{-1} W^*(0) (\tilde{F} - \tilde{F})^{-1} W(0) (\tilde{F} W(0))^{-1} > 0.$$ 

The last inequality follows from the assumption on the sign of $\varphi_j / \tilde{\varphi}_j > 0$ for all $j = 1, \ldots, m$. We obtain that

$$\lim_{\lambda, \sigma \to 0} U(\lambda, \sigma) = \frac{1}{2} \left[ ((\tilde{F} - \tilde{F})^{-1} + H^* \tilde{F}(0)) (I + (\tilde{F} - \tilde{F}) H(0)) + (I + H^* \tilde{F}(0)(\tilde{F} - \tilde{F}))((\tilde{F} - \tilde{F})^{-1} + H^* \tilde{F}(0)) \right] > 0.$$ 

From now on, we proceed as previously, starting from the existence of $v_0$ such that $v_0 = -\tilde{F} H(0) v_0$; hence,

$$v_0^* U(0, 0) v_0 = v_0^* (H(0) (I - \tilde{F}^* \tilde{F})^{-1}) v_0 \sum_j \left( \frac{1}{\varphi_j} - \frac{1}{\tilde{\varphi}_j} \right) \left( 1 - \frac{\varphi_j}{\tilde{\varphi}_j} \right) (v_0^j)^2 < 0,$$

where we again considered $v_0^j = 0$—those entries of $v_0$ that correspond to those $\tilde{\varphi}_j$, entries of $\tilde{F}$, that are 0. This gives again a contradiction in the properties of $U(0, 0)$ and ends the entire proof concerning exponential stability of the linear system (99) for all $\tilde{F}$ with the entries satisfying (98). But this means nothing more but system (23) being exponentially stable for all linear functions $\varphi_j(v)$ satisfying the sector and the slope restrictions.

\footnote{In fact, the existence of the stabilizing linear feedback defined by $\tilde{F}$ and the frequency domain inequality (27) ensure, according to, for example, the hyperstability Theorem 1 of [10, Section 16], that $T(\sigma, \sigma)$ is positive real, which gives $r \leq 1$ and accepts $r = 2$ provided $D_1 = 0$ (see also [53] on the so-called limit stability).}
In this way, the properties of $\mathcal{W}^\varphi_F(x)$ and $\mathcal{W}^\varphi_D(x)$, which were valid for $F = \tilde{F}$, are valid for all $F$ whose entries are subject to (98), that is, to (24). Both $\mathcal{W}^\varphi_F(x)$ defined by (100) and $\mathcal{W}^\varphi_D(x) = \lim_{\epsilon \to 0} \mathcal{W}^\varphi_F(x)$ are positive definite for all $F$ in the parallelepiped defined by (98). This set is compact; hence,

$$\mathcal{W}^\varphi_F(x) \geq \delta_0 |x|^2, \quad \mathcal{W}^\varphi_D(x) \geq \delta_0 |x|^2,$$

(108)

where $\delta_0 > 0$ is independent of the choice of $F$. We follow now [38] and use a technique by I. G. Malkin [54, Chapter 3, Section 26]: take in $\mathcal{W}^\varphi_F(x)$, for arbitrary $x$,

$$\gamma_j = \frac{\varphi_j(c_j^* x)}{c_j^* x}, \quad c_j^* x \neq 0; \quad \frac{\varphi_j(c_j^* x)}{c_j^* x} = 0, \quad c_j^* x = 0, \quad j = 1, m,$$

(109)

we dropped the superscript for $\mathcal{W}_F$ because the estimate (108) is the same, as well as the function structure. We deduce from (100) in both cases

$$\mathcal{W}_F(x) = -x^*N_{11}x + f^*(C^* x)(N_{12})^*x + x^*N_{12}f(C^* x) - f^*(C^* x)N_{22}f(C^* x)$$

$$+ \frac{1}{2} \sum_{i=1}^m (\theta_j^i - \theta_j^i/\varphi_j^i)(c_j^* x)\varphi_j(c_j^* x)$$

$$= \gamma(x) - \sum_{i=1}^m (\theta_j^i - \theta_j^i/\varphi_j^i) \left( \int_0^{c_j^* x} \varphi_j(\lambda) d\lambda - \frac{1}{2}(c_j^* x)\varphi_j(c_j^* x) \right).$$

(110)

With (108) and (110) being combined, it follows that

$$\gamma(x) \geq \delta_0 |x|^2 + \sum_{i=1}^m (\theta_j^i - \theta_j^i/\varphi_j^i) \left( \int_0^{c_j^* x} \varphi_j(\lambda) d\lambda - \frac{1}{2}(c_j^* x)\varphi_j(c_j^* x) \right),$$

(111)

According to the assumption on the asymptotic behavior of $\varphi_j(v)$, it follows that $\lim_{|x| \to \infty} \gamma(x) = +\infty$. We may thus apply the Liapunov-like Lemma 1 (Lemma 2.3.2 of [44]) to obtain global asymptotic stability (the fact that the stationary set of (23) is the singleton \{0\}, that is, the equilibrium at the origin, has been also taken into account).

It is worth mentioning that the properties of the Liapunov function $\gamma(x)$ are, in fact, stronger than the requirements of the Lemma. This may be useful in obtaining other qualitative properties, for example, forced oscillations and other.

### 6. CONCLUSIONS AND PERSPECTIVES

There are several aspects to be pointed out starting from the theoretical and applied results of this paper. The stability criterion of Theorem 1 is concerned with systems containing several sector-restricted and slope-restricted nonlinearities. For such systems, the specific features of the frequency domain stability inequality follow from a quite large variety of stability multipliers. Our choice was directed to the Yakubovich-type multipliers, which are anticipative (noncausal) and have been introduced under various conditions by many researchers. The studied system was, however, the same: it contained a single nonlinear element, and the linear subsystem was exponentially stable. According to the mathematical technicalities, different technical assumptions have been made.

It was established in this paper that the oldest approach of Yakubovich [19, 20] allows obtaining the stability criterion of a given class with a minimum of technical assumptions. The only drawback is that the slope restrictions may be introduced only in conjunction with the sector ones.

On the other hand, the technical problem of PIO—pilot-in-the-loop oscillations of aircrafts—or more precisely the so-called PIO II (where all elements of the feedback system airframe pilot are linear except the position and rate limiters), suggested a renewed interest for the absolute stability problem. Moreover, the saturation nonlinearity that is contained in the limiters is both sector and slope restricted. At the same time, the linear part of the PIO II theoretical structure is either in critical or in the unstable case.
This motivated the approach of the paper—to consider, in the unified way promoted by V. M. Popov [10], the stable, critical, and unstable cases for the linear block of the absolute stability structure of Figure 1, also the multi-input case, that is, systems with several nonlinear elements. This speaks for the theoretical novelty of the main result enclosed in Theorem 1. The method of proof is a certain way of combining the a priori information on the nonlinear elements summarized in the (integral) quadratic constraints. At their turn, these constraints allow the construction of the integral index from the definition of the so-called Popov system. Further, the frequency domain inequality allows the construction of a Liapunov function having its derivative along the system’s solutions of at least nonpositive definite. This Liapunov function turns to be the most refined instrument of proof.

In the applications, we followed the ‘classical’ way that makes of the frequency domain inequality an instrument of practical stability, not of theoretical proof only. This is because, in the PIO problem, the databases contain the information in the language of the frequency domain.

The applications analyzed in the paper, using classical analytical techniques, emphasized some interesting conclusions: increasing complexity of the frequency domain inequality surely increases the manipulation difficulties, but it does not always increase the absolute stability domain in the parameter space; this motivates the ‘parsimony principle’ stated in the paper.

The results of the paper may be extended to other cases. Firstly, it is not without interest to apply the unified approach for stable, critical, and unstable cases in order to obtain frequency domain stability inequalities containing other stability multipliers. Another possible extension would be to time delay and distributed parameter systems. Here, the approach of integral equations seems more adequate. Incorporating at least the critical cases within this approach, for slope-restricted nonlinearities, may be an immediate perspective, if we take into account, for example, the time delay in the pilot models for the PIO problem.

One should add to the possible development the cases discussed in [39, 40] (especially taking into account the appealing applications and the discontinuous right-hand side). Improvement of the criterion for the case of slope restrictions only [30] is also a possible task for the future. Here, the research is in progress, including a generalization of [35] to the multivariable case.

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