The thesis contains the results of the author and is organized in two parts. The first part entitled *The geometry of Lie algebroids* has five chapters and contains the mathematical apparatus and framework for the second part, where we study the optimal control problems with economic applications, using the Pontryagin Maximum Principle. Differential geometric methods have the advantage of providing global results, rather than local results, since they are independent of the choice of coordinates. In the first chapter some preliminary results concerning geometrical structures on Lie algebroids are presented. The notion of prolongation of a Lie algebroid over the vector bundle projection is studied in the chapter two. We focus on the second order differential equations (SODE) which can be represented using the notion of semispray. The Ehresmann nonlinear connection induced by SODE is investigated and its tensors of torsion and curvature are studied. Also, the autoparallel curves (geodesics) of nonlinear connection are pointed out, and these are optimal solutions of the control problems studied in the second part. Given a regular Lagrange function, we find the semispray and the canonical nonlinear connection which depends on Lagrangian and structure functions of the Lie algebroid. We also have the Euler-Lagrange equations on Lie algebroids.

In the chapter three we introduce the notion of dynamical covariant derivative and metric nonlinear connection at the level of Lie algebroids $TE$. We prove that the canonical nonlinear connection induced by a regular Lagrangian is a unique connection which is metric and compatible with the symplectic structure induced by the regular Lagrangian. Also, the inverse problem of variational calculus on Lie algebroids is investigated and the Helmholtz conditions are given, because it is important to know if the second order differential equations come from a variational principle. Moreover, because the Euler-Lagrange equations are second order differential equations, generally very difficult to solve, we transfer the problems to the dual space, equipped with a regular Hamiltonian function.
via Legendre transformation. However, if the geodesics (the optimal trajectories of control systems) are situated on a manifold with positive constant curvature, then the geodesics focus at certain points, and the negative curvature disperses geodesics, these never intersect again.

In the chapter four we deal with the prolongation of a Lie algebroid over the vector bundle projection of a dual bundle. We introduce the notions of dual adapted tangent structure and regular section. These structures induce a canonical nonlinear connection. In the case of Hamiltonian formalism we have the corresponding Hamilton-Jacobi equations on Lie algebroids. Also, the duality between the Lagrangian and Hamiltonian formalisms on Lie algebroids is proved, which justify the methods applied in control theory. In the chapter five we introduce the notion of dynamical covariant derivative and metric nonlinear connection at the level of Lie algebroids $\mathcal{T}E^*$. We prove that the canonical nonlinear connection induces by a regular Hamiltonian is the unique metric and symmetric nonlinear connection.

The purpose of the second part entitled Lie geometric methods in optimal control is to study the driftless control-affine systems (distributional systems) with positive homogeneous cost, using the Pontryagin Maximum Principle at the level of a Lie algebroid in the case of constant rank of distribution. An optimal control system is a set of differential equations (constraints) describing the paths of the control variables that minimize the cost function (energy, cost, time, distance, etc). A driftless control-affine system (or distributional systems) has the form

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

The vector fields $X_i$, $i = 1, m$, generate a distribution $\Delta$ on the manifold $M$ such that the rank of $\Delta$ is constant. Let $x_0$ and $x_1$ be two points of $M$. An optimal control problem consists of finding those trajectories of the distributional system which connect $x_0$ and $x_1$, while minimizing the cost

$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t)) dt,$$

where $\mathcal{F}$ is a positive homogeneous function. We prove that the framework of Lie algebroid is more useful than cotangent bundle in order to solve some problems of driftless control-affine systems. In the first chapter, the known results on the optimal control systems are recalled by geometric viewpoint and the Pontryagin Maximum Principle in the framework of the cotangent bundle and the dual Lie
algebroid is presented. In fact, the optimal control is a generalization of the
variational calculus and the Pontryagin’s principle leads to the Hamilton-Jacobi-
Bellman equations together with a maximization condition for the Hamiltonian
with respect to the control variables $u(t)$. In the second chapter the control-affine
systems are presented and the relation between the Hamiltonian functions on $E^*$
and $T^*M$ is given.

Also, the problems of controllability in terms of Lie brackets are pointed out
and the theorems of Frobenius for integrable distribution and Chow-Rashevsky
for bracket generating distribution are presented. Controllability is the ability to
steer a system from a given initial state to any final state, in finite time, using the
available controls. If the distribution $\Delta = \text{span}\{X_1, ..., X_m\}$ is bracket generating
(nonholonomic), then the driftless control-affine system is controllable. If $\Delta$ is
integrable (holonomic) then the system is not controllable and $\Delta$ determines a
foliation on $M$ with the property that any trajectory (solution of the control
system) is contained in a single leaf of the foliation, and the restriction of $\Delta$ to
each leaf of the foliation is bracket generating. With the other words, two points
can be joined if and only if they are situated on the same leaf.

In the chapter three, we investigate the cases of holonomic and nonholonomic
distributions with constant rank. In the holonomic case, we will consider the
Lie algebroid being just the distribution, whereas in the nonholonomic case (i.e.,
strong bracket generating distribution $\text{rank}\Delta = 2$) the Lie algebroid is the tangent
bundle with the basis given by vectors of distribution, completed by the first
Lie brackets. More interesting for its applications is the case of nonholonomic
distributions and these systems arise in kinematic models of many control systems.
We present, as practical applications, the mathematical model of maneuvering a
car (Dubins shortest path problem or minimum-time problem) and a differential-
drive robot, where the state spaces are Lie algebroids.

The case of distribution $\Delta$ with non-constant rank is studied in the chapter
four and some interesting examples are given. In the chapter five we present the
intrinsic relation between the distributional systems and sub-Riemannian geometry.
Thus, the optimal trajectories of our distributional systems are the geodesics
in the framework of sub-Riemannian geometry. We investigate two classical cases:
Grusin plan and Heisenberg group, but equipped with positive homogeneous costs.
We are using the Pontryagin Maximum Principle at the level of Lie algebroids in
the case of Heisenberg group and show that this idea is very useful in order to
solve a large class of distributional systems. In the last chapter, the conclusions
and further developments are presented.
However, we cannot always use the framework of Lie algebroids in the study of driftless control affine systems, see for instance the case of non-constant rank of distribution or bracket generating distributions with the $\text{rank} \Delta \geq 3$. Also, the control affine systems with drift and economic models where the constraints are not linear will be considered for the study in the future. In this category is included, for example, a simple Lotka-Volterra system that models a population of a harvested prey in the presence of a predator. In particular, we will study the planar control-affine systems, and the problem of reaching every point of the plane in minimum time starting from the origin. Moreover, we propose the study of the second order driftless control affine systems and the stochastic optimal control that deals with the existence of uncertainty either in observations or in the evolution of the system. The thesis contains the final list of 95 cited references.