University of Craiova
Doctoral School of Sciences
Physics

Summary

PhD Thesis

Integrability and classes of solutions for nonlinear evolution equations

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Craiova
2022

Contents

The doctoral dissertation entitled "Integrability and classes of solutions for nonlinear evolution equations" sets out some methods for analyzing the integrability of nonlinear systems, as well as several approaches for determining certain types of solutions of equations that describe their evolutions. Numerical approximations are not precise enough and it is very difficult to formulate a general theory of solving nonlinear equations, so there are different methods for solving them. The diversity and multiplicity of solving methods result from the particularities of each equation. The order of nonlinearity, dispersion, dissipation, all these play an important role in searching and finding of solutions or in determining the integrability of the considered system. As the principle of superposition is no longer valid, the existence and uniqueness of solutions are not guaranteed.

Depending on the characteristics of each method, they are classified in this doctoral study in two broad categories:

-direct solving methods that can generate several classes of solutions: soliton solutions, traveling wave solutions, singular and periodic solutions, solutions expressed by means of elliptic functions;

- rigorous methods for the exhaustive study of nonlinear systems, which provide information regarding the integrability, but also on the special solutions, of soliton or multi-soliton type, which the investigated system admits. From this category of methods we will present the Inverse Scattering Method, the Lax Pair method and the Dressing Method.

The paper is structured in seven sections.

After the **first section**, which is the introduction of the paper, in the **second section** there is a presentation of the general notions of nonlinear dynamics, basic, elementary concepts, such as general aspects of the theory of differential equations, stability and equilibrium points, dynamic systems with chaotic behavior.

The **third part,** which is consistent in terms of original contributions, aims to investigate the main models with nonlinear dynamics, presenting methods of direct solution and classes of solutions of the equations attached to these models. Assuming that a partial differential equation can be transformed into an ordinary differential equation by introducing the wave transformation $\xi = kx \pm \lambda t$, $u(x,t) = U(\xi)$ where k is the wave number and λ represents the wave velocity, several solving methods have been studied to obtain the solutions of the considered systems. The structures of this part and the main contributions are:

3.1.The attached flow method. The first paragraph of this part is describing the Attached Flow Method, which assures us, in the case of the studied models, that their solutions can be expressed in terms of the solutions of other equations. The method consists in supplementing the equation to be studied with a first order equation, of the type of flow equations in the case of dynamical systems, in which the first order derivative of the dependent variable$U\prime(\xi)$ is expressed by means of a function $V(U)$, identified as the flow attached to the variable. Solving the equation initially considered is reduced to solving an equation with a new independent variable on $V(U)$. The advantage of this method is that it reduces the order of the equation resulting from the introduction of the wave variable. Although the solutions found are particular, due to the constraint $U\prime = V(U)$, the method offers traveling wave or soliton solutions. This method represents the author contribution, proposed following the difficulties encountered in finding solutions to difficult equations, such as the Fisher and Gardner equations, by alternative methods.

Using the attached flow method we obtained particular solutions for important nonlinear models.

The Benjamin-Bona-Mahony equation describes the unidirectional propagation of long waves, with small amplitude, from the water surface, but also the propagation of acoustic waves:

$$u_t - u_{xxt} + u_x(1 + u^n) = 0. \tag{0.1}$$

For $n = 1$ the solution of the BBM equation is:

$$u(x,t) = \frac{3(\lambda - 1)}{\cosh^2(\frac{1}{2\lambda}(x - \lambda t)\sqrt{\lambda(\lambda - 1)})}. \tag{0.2}$$

The graphical representation of the solution is in Fig .10:

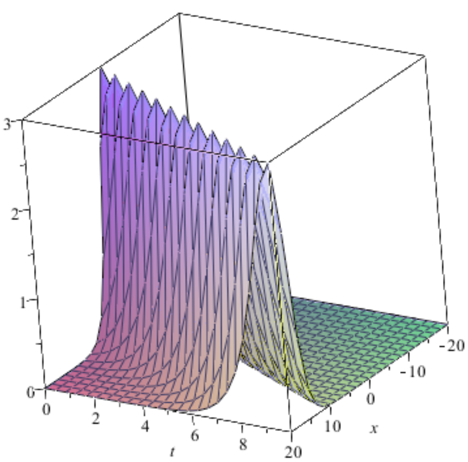

Fig. 10. The solution of the BBM equation for $\lambda = 1$.

For $n = 2$ the solution of the BBM equation is:

$$u(x,t) = \frac{24\lambda(\lambda - 1)\,(\cosh A + \sinh A)}{\cosh 2A + \sinh 2A + 24\lambda^2(\lambda - 1)}, \tag{0.3}$$

where:

$$A = \frac{(x - \lambda t)\sqrt{\lambda - 1}}{\sqrt{\lambda}} \tag{0.4}$$

For $n = 4$ the solution of the BBM equation is:

$$u(x,t) = -\frac{2\sqrt{15\lambda(\lambda - 1)}\,(\cosh A + \sinh A)}{\sqrt{\cosh 4A + \sinh 4A + 60\lambda^2(\lambda - 1)}}, \tag{0.5}$$

The graphical representation of the solution is in Fig. 11a for $\lambda = 2$, Fig. 11b for $\lambda = 3$:

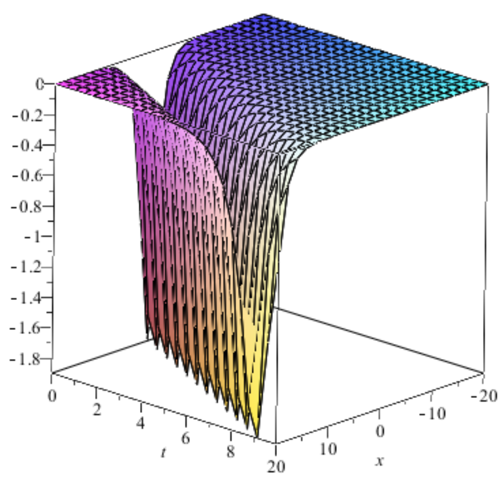

Fig. 11a

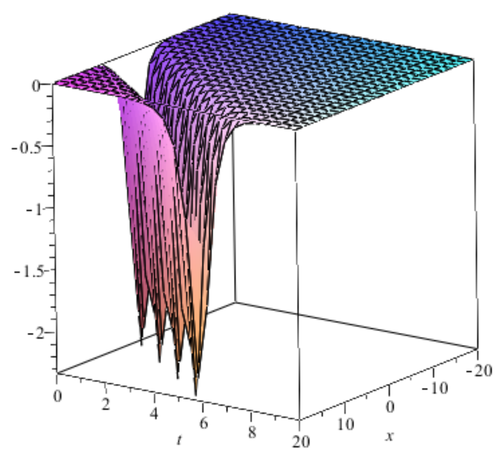

Fig. 11b

The Fisher equation with applications in heat and mass transfer, flame propagation, Brownian motion processes, even in nuclear reactor theory is:

$$u_t = \alpha u_{xx} + \beta u(1 - u). \tag{0.6}$$

Using the attached flow method we get the solution:

$$u(x,t) = \frac{(\cosh A - \sinh A)(3\cosh A - \sinh A)}{4\cosh^2 A}, \tag{0.7}$$

where:

$$A = \frac{\sqrt{-6\alpha\beta}x - 5\alpha\beta t}{12\alpha}. \tag{0.8}$$

The Gardner equation models the propagation of acoustic ions in plasma:

$$u_t + u_{xxx} + 2\alpha u u_x - 3\beta u^2 u_x = 0. \tag{0.9}$$

has the solution:

$$u(x,t) = \frac{144\lambda\left(\cosh A - \sinh A\right)}{-2592\lambda\beta + 576\alpha^2 + 48\alpha\left(\cosh A - \sinh A\right) + \left(\cosh 2A - \sinh 2A\right)}, \tag{0.10}$$

where $A = (x - \lambda t)\sqrt{\lambda}$.

The Ricci flow equation:)

$$u_t = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2}. \tag{0.11}$$

has the solution:

$$u(x,y,t) = \frac{k}{\sinh\left(\frac{\lambda k(\alpha x+\beta y-\lambda t)}{2\alpha\beta}\right) + \cosh\left(\frac{\lambda k(\alpha x+\beta y-\lambda t)}{2\alpha\beta}\right) - 2}. \tag{0.12}$$

The graphical representation of the solution is in Fig 12.

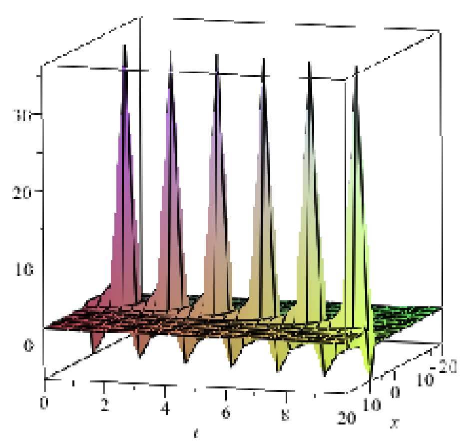

Fig. 12. The solution of Ricci equation
for $\alpha = 1/5$, $\beta = -1$, $\lambda = 1$, $k = 0, 5$,
$y = 0$.

The Ţiţeica equation initially appeared in the field of geometry, proposed by G. Ţiţeica, with numerous applications in quantum field theory:

$$u_{xt} = e^u - e^{-2u}. \tag{0.13}$$

The solution of the Ţiţeica equation is (Fig. 13):

$$u(x,t) = -\ln(2) - 2\ln\left(\cosh\frac{(x-\lambda t)\sqrt{3}}{2\sqrt{\lambda}}\right) + \ln\left(3 - 2\cosh^2\frac{(x-\lambda t)\sqrt{3}}{2\sqrt{\lambda}}\right). \qquad (0.14)$$

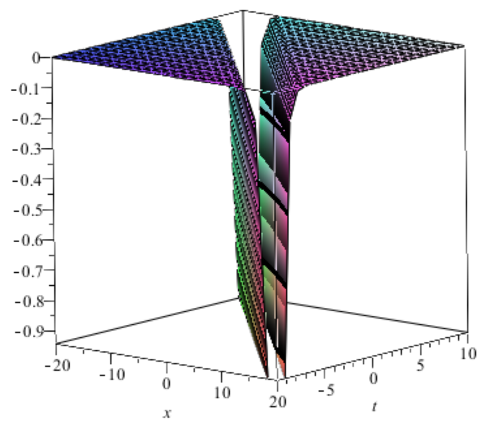

Fig. 13 The solution of the Titeica equation.

Whitham - Broer - Kaup system (WBK) model describes the dispersive waves from the water surface:

$$u_t + uu_x + h_x + \beta u_{2x} = 0 \qquad (0.15)$$

$$h_t + (hu)_x + \alpha u_{3x} - \beta h_{2x} = 0 \qquad (0.16)$$

has the solutions (Fig. 14):

$$\begin{aligned} u(x,t) &= -\frac{2\lambda(\cosh A + \sinh A)}{\cosh B + \sinh B - (\cosh A + \sinh A)} \\ h(x,t) &= -\frac{2\lambda^2(\sqrt{\beta^2+\alpha}+\beta)(\cosh C + \sinh C)}{\sqrt{\beta^2+\alpha}(\cosh B + \sinh B - \cosh A - \sinh A)^2} \end{aligned} \qquad (0.17)$$

where $A = \frac{\lambda^2 t}{\sqrt{\beta^2+\alpha}}$, $B = \frac{\lambda x}{\sqrt{\beta^2+\alpha}}$, $C = \frac{\lambda(x+\lambda t)}{\sqrt{\beta^2+\alpha}}$.

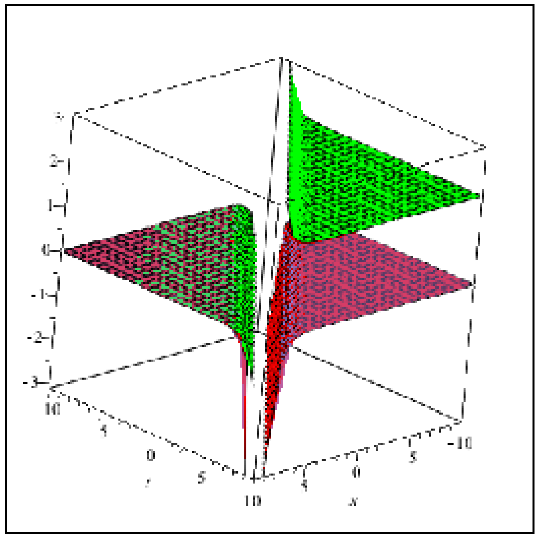

Fig. 14. The solutions of the WBK
system for $\alpha = 0,2$, $\beta = 0,66$, $\lambda = 1$.

From the attached flow method we observe that imposing the condition $U' = V(U)$ does nothing but express the solutions of the investigated equations in terms of the solution of the equation resulting from the discovery of the specific form of $V(U)$. The results were published in indexed journals [P 1],[P 2], [P6].

3.3 The Axiliary Equation Technique was included as follows:

3.3.1 Riccati equation as an auxiliary equation. A method, often used in the literature, for searching for traveling wave solutions, is represented by the Hyperbolic Tangent Method which requires searching for solutions in the form of predefined expressions, constructed using this function $Z = \tanh(\mu\xi)$. This method helps us to express the solution of the ordinary differential equation, resulting from the introduction of the wave transformation, in an expansion of the hyperbolic tangent $U(\xi) = \sum_{i=0}^{m} a_i Z^i$ .Where $m > 0$ is an integer, which is determined by making the balance between the highest order of the derivative and the highest order of nonlinearity of the considered equation [17].

The description of this method was made only to support the presentation of the solving methods that have in common the use of auxiliary equations, which are found in the third paragraph of section 3.

This idea of using the auxiliary equations as a solving method came from the fact that the Riccati equation $G' = k + G^2$ admits the hyperbolic tangent as a fundamental solution. This is another approach in which the solutions of nonlinear equations are expressed in terms of the solutions of another equation, which can be solved and which is called the auxiliary equation. Using the Riccati equation as an auxiliary equation, we obtained solutions for the model that describes the propagation of sound in cylindrical biological membranes, similar to the equation that describes the propagation of nerve pulses through neurons, a generalized variant of the Boussinesq equation:

$$u_{2t} - A'(u)u_x^2 - A(u)u_{2x} + hu_{4x} = 0. \tag{0.18}$$

with:

$$A(u) = au^2 + bu + \alpha. \tag{0.19}$$

For $a \neq 0$ we have :

$$u(x,t) = -\frac{b}{2a} - \sqrt{\frac{6h}{a}}\sqrt{-\frac{-b^2-\lambda^2 a+4\alpha a}{8ah}}\tanh\sqrt{-\frac{-b^2-4\lambda^2 a+4\alpha a}{8ah}}(x+\lambda t),$$

For $a = 0$ the solution is:

$$u(x,t) = -\frac{\alpha-\lambda^2-8hk}{b} + \frac{12h}{b}k\tanh^2(\sqrt{-k}x+\sqrt{-k}\lambda t).$$

3.3.2. The G'/G method. Using the G'/G method for the same generalized model $u_{2t} - A'(u)u_x^2 - A(u)u_{2x} + hu_{4x} = 0$, we extended the family of solutions of the considered equation. This method, in addition to appealing to an auxiliary equation of form $G' = r + pG + qG^2$, it develops the solutions of the studied equation in terms of a relation between the derivative of the solution and the solution of the auxiliary equation considered $U(\xi) = \sum_{i=1}^{n} d_i(G'/G)^i$ [11].

a) For $p^2 - 4qr > 0$, $pq \neq 0$, $qr \neq 0$, the soltion of the auxiliary equation is $G_1(\xi) = -\frac{\sqrt{p^2-4qr}\tanh(\frac{1}{2}\sqrt{p^2-4qr}\xi)}{2q}$, thus for the considered equation we have the solution:

$$u_1(x,t) = -\frac{b}{2a} - \frac{2rq\sqrt{6ah}}{a(p+B\tanh(\frac{1}{2}B(x-\frac{1}{2}\sqrt{A}t))} - \frac{\sqrt{6ah}B\tanh(\frac{1}{2}B(x-\frac{1}{2}\sqrt{A}t)}{2a}$$

where $A = 4\alpha + 2hp^2 + 16hqr - \frac{b^2}{a}$, $B = \sqrt{p^2-4qr}$.

b) For $G_2(\xi) = \frac{2r\cosh(\frac{1}{2}\sqrt{p^2-4qr}\xi)}{\sqrt{p^2-4qr}\sinh(\frac{1}{2}\sqrt{p^2-4qr}\xi)-p\cosh(\frac{1}{2}\sqrt{p^2-4qr}\xi)}$ we have:

$$\begin{aligned} u_2(x,t) &= -\frac{b}{2a} + \frac{\sqrt{6ah}B\sinh(\frac{1}{2}B(x-\lambda t))}{2a\cosh(\frac{1}{2}B(x-\lambda t))} \\ &\quad + \frac{2r\sqrt{6ah}\cosh(\frac{1}{2}B(x-\lambda t))}{a(\sqrt{p^2-4qr}\sinh(\frac{1}{2}B(x-\lambda t))-p\cosh(\frac{1}{2}B(x-\lambda t))}. \end{aligned}$$

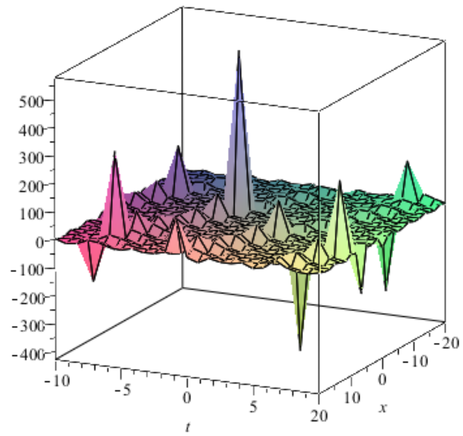

Fig. 15 Solution obtained for the parametric values $q = 2$, $p = 0,5$, $r = 0,25$, $a = 1$, $b = 0,125$, $h = 1$, $\alpha = 0,66$.

c) For $G_3(\xi) = \frac{2r\sinh(\sqrt{p^2-4qr}\xi)}{\sqrt{p^2-4qr}-p\sinh(\sqrt{p^2-4qr}\xi)-\sqrt{p^2-4qr}\cosh(\sqrt{p^2-4qr}\xi)}$ we have the following solution:

$$
\begin{aligned}
u_3(x,t) &= -\frac{b}{2a} - \frac{\sqrt{6ah}B\cosh(B(x-\lambda t))}{2a\sinh(B(x-\lambda t))} - \frac{\sqrt{6ah}B}{2a\sinh(\lambda t - x)} \\
&+ \frac{2rq\sqrt{6ah}\sinh(B(x-\lambda t))}{a(-p\sinh(B(x-\lambda t)) - \cosh B(x-\lambda t))B + B)},
\end{aligned}
$$

unde $B = \sqrt{p^2 - 4qr}$.

d) For $p^2 - 4qr < 0$, $pq \neq 0$, $qr \neq 0$, the soltion of the auxiliary equation is $G_4(\xi) = -\frac{1}{2q}(p + \sqrt{4qr-p^2}\cot(\frac{\sqrt{4qr-p^2}}{2}\xi)$ thus we have the following solution:

$$
u_4(x,t) = -\frac{b}{2a} - \frac{2rq\sqrt{6ah}}{a(p + C\cot(\frac{1}{2}C(x-\lambda t))} - \frac{\sqrt{6ah}C\cot(\frac{1}{2}C(x-\lambda t)}{2a}
$$

where $C = \sqrt{4qr - p^2}$.

We notice that by appealing to an auxiliary equation of a higher order, as well as to a more complex development in terms of the solution of the respective equation, we can generalize families of solutions of the equations we want to investigate. Depending on the relationships between the parameters involved in the Riccati equation, there are over 27 solutions in the literature, in terms of which we can obtain new solutions for the generalized version of the Boussinesq equation.

3.3.3. The Extended Trial Equation Method (ETEM). The ETEM was applied to the LS (Long Short wave resonance model). This method uses a general auxiliary equation $(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_p\Gamma^p+...\xi_1\Gamma+\xi_0}{\nu_n\Gamma^n+....\nu_1\Gamma+\nu_0}$, which after imposing some constraints for the compatibility with the analyzed system is of the form $\frac{\Phi'}{2\Psi} = \frac{a_1\left(4\xi_4\Gamma^3+3\xi_3\Gamma^2+2\xi_2\Gamma+\xi_1\right)}{2\nu_0}$. The LS wave model is described by the following system of equations:

$$\begin{aligned} iS_t + \alpha S_{xx} - Ls &= 0, \\ L_t + \beta\left(|S|^2\right)_x &= 0, \end{aligned} \tag{0.20}$$

and one of the solutions obtained using the extended test equation method:

$$S_4(t,x) = \exp\left(i\left(\mu_1 x + \mu_2 t\right)\right)\frac{K_2}{P+\cosh\left[R\left(\rho_1 x+\rho_2 t\right)\right]}, \tag{0.21}$$

$$L_4(t,x) = \frac{1}{2\mu_1}\left[\frac{K_2}{P+\cosh\left[R\left(\rho_1 x+\rho_2 t\right)\right]}\right], \tag{0.22}$$

where $K_2 = \frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)a_1}{\alpha_3-\alpha_2}$, $p = \frac{2\alpha_1-\alpha_2-\alpha_3}{\alpha_3-\alpha_2} \neq 0$, $R = \frac{\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}}{K}$.

3.3.4. The Kudryashov method. A generalization of the Kudryashov method was used for the same LS model, using as an auxiliary equation $Q' = Q^2 - Q$ (the Kudryashov equation). One of the solutions obtained using the Kudryashov method is:

$$S_9(t,x) = -\exp\left(i\left(\mu_1 x+\mu_2 t\right)\right)\frac{a_2}{2}\tanh\left(\frac{\rho_1 x+\rho_2 t}{2}\right), \tag{0.23}$$

$$L_9(t,x) = \frac{a_2^2}{8\mu_1}\left[\tanh\left(\frac{\rho_1 x+\rho_2 t}{2}\right)\right]^2. \tag{0.24}$$

The analysis of the LS model was performed using a new perspective that led to a classification of exact solutions by two methods, the ETEM method and the Kudryashov method. Thus, dark and bright solitons were highlighted, solutions expressed with the help of rational functions, singular and periodic waves. In terms of the solutions of these two auxiliary equations, new solutions were obtained, reported in a paper published in the journal Open Physics [ISI 1]

3.3.5. The Jacobi elliptic equation as an auxiliary equation. Next we considered the Bullough-Dodd model for which we used the Jacobi elliptic equation as an auxiliary equation $(\phi')^2 = h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi)$ which under certain conditions degenerates into several equations involving a low number of parameters. Using this subset of equations we have built a new set of solutions for the Bullough-Dodd (BD) model with wide applications in fluid dynamics, nonlinear optics or solid state physics. The results are based on the main idea of finding the relationships between the 12 parameters that occur during the application of the method and were published in the journal Symmetry [ISI 2].

Starting from the general equation:

$$u_{xt} = -pe^{u} - qe^{-2u} \tag{0.25}$$

and using the transformation $u = \ln v$ we found the solutions of the equation:

$$vv_{xt} - v_x v_t + pv^3 + q = 0, \tag{0.26}$$

in terms of the solutions of the Jacobi elliptic equation. For $h_0 = h_1 = 0$, $h_2 = 4$, $h_3 = -\frac{4(2\beta+\rho)}{\alpha}$, $h_4 = \frac{\gamma^2+4\beta^2+4\beta\rho}{\alpha^2}$, the hyperbolic solutions that depend on 6 parameters are:

$$v(x,t) = a\left[1 - \frac{3\left(\gamma^2+4\beta^2\right)}{8\beta}\frac{\left[\sec h\left(kx+\lambda t\right)\right]^2}{\beta\left[\sec h\left(kx+\lambda t\right)\right]^2 + \gamma\tanh\left(kx+\lambda t\right) - \frac{\gamma^2+4\beta^2}{8\beta}}\right]. \tag{0.27}$$

In the **fourth part** were presented notions about the inverse scattering problem, Lax representation and spectral analysis of Lax operators, notions regarding the analysis of the integrability of nonlinear evolution equations. And in the **fifth part**, original results were obtained, following the collaboration with Prof. Vladimir Gerdjikov, from the Institute of Nuclear Research and Nuclear Energy and the Institute of Mathematics and Informatics in Sofia, Bulgaria. We considered a class of Lax operators related to the symmetric spaces BD.I, with the help of which we solved the class of nonlinear vector equations with applications in the Bose-Einstein condensate, respectively the Kulish-Sklyanin (KS) model. In the second paragraph of this section, using the Zakharov-Shabat dressing method, 1-soliton type solutions were generated, using rank 1 and rank 2 projectors, but the explicit form whose internal structure is described by the projector rank 2 is very complicated. In the sixth paragraph of this part a rank 2 general projector with a convenient parameterization of polarization vectors was chosen. We have constructed the kernel of the solvent that determines the spectrum of the Lax operator and we have demonstrated that the fundamental analytical solutions satisfy the completeness relations.

The Inverse scattering problem (ISP) is perceived as an equivalent approach to the Riemann-Hilbert problem [23],[24].

In [22] Kulish and Sklyanin discovered a class of Schrodinger nonlinear vector equations:

$$i\vec{q}_t + \vec{q}_{xx} + 2\left(\vec{q}^{\dagger}, \vec{q}\right)\vec{q}(x,t) - \left(\vec{q}, s_0\vec{q}\right)s_0\vec{q}^{*}(x,t) = 0, \tag{0.28}$$

where $\vec{q}(x,t)$ is a $2r-1$ components vector function that tend to zero for $|x| \to \infty$, and s_0

is a constant matrix:

$$S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (E_{kn})_{ij} = \delta_{ik}\delta_{nj}, \tag{0.29}$$

where the condition of orthogonality is given by the relation:

$$X \in SO(2r+1) \text{ dacă si numai dacă } X + S_0 X^T S_0 = 0. \tag{0.30}$$

The KS model has been extensively studied due to its applications in Bose-Einstein Condensate (BEC). The equations allow Lax representation:

$$[L(\lambda), M(\lambda)] = 0, \tag{0.31}$$

where the operators are given by:

$$L\psi(x,t,\lambda) \equiv i\partial_x \psi + (Q(x,t) - \lambda J)\,\psi(x,t,\lambda) = 0, \tag{0.32}$$
$$M\psi(x,t,\lambda) \equiv i\partial_t \psi + \left(V_0(x,t) - \lambda V_1(x,t) - \lambda^2 J\right)\psi(x,t,\lambda) = 0, \tag{0.33}$$

The Lax operator's Jost solutions are defined by their asymptotes for $x \to \pm\infty$:

$$\lim_{x\to+\infty} \psi(x,t,\lambda)\, e^{i\lambda J x} = \mathbf{1}, \tag{0.34}$$
$$\lim_{x\to-\infty} \phi(x,t,\lambda)\, e^{i\lambda J x} = \mathbf{1}. \tag{0.35}$$

In the case of the class of equations corresponding to symmetric spaces of $BD.I$ type (0.28) we chose $(2r+1)\times(2r+1)$ block matrix structure:

$$Q(x,t) = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{q}^* & 0 & s_0\vec{q} \\ 0 & \vec{q}^\dagger s_0 & 0 \end{pmatrix}, \qquad J = \mathrm{diag}(1,0,...,0,-1) \tag{0.36}$$

The choice of the J matrix determines the structure of the symmetric space. Given the choice of the J matrix, as well as the fact that the solutions Jost and the matrix $T(t,\lambda) = \psi^{-1}(x,t,\lambda)\,\phi(x,t,\lambda)$, belong to the $SO(2r+1)$ group, it can be used the following structure of the scattering matrix and its inverse $\hat{T}(\lambda,t)$:

$$T(\lambda,t) = \begin{pmatrix} m_1^+ & -\vec{b}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{B}^- \\ c_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{pmatrix}, \quad \hat{T}(\lambda,t) = \begin{pmatrix} m_1^- & \vec{B}^{-T} & c_1^- \\ -\vec{B}^+ & \hat{\mathbf{T}}_{22} & s_0\vec{b}^- \\ c_1^+ & -\vec{b}^{+T}s_0 & m_1^+ \end{pmatrix}, \tag{0.37}$$

where $\vec{b}^\pm(\lambda,t) = \mathbf{b}^\pm(\lambda,t)$ şi $\vec{B}^\pm(\lambda,t) = \mathbf{B}^\pm(\lambda,t)$ are $(2r-1)$ component vectors, $\mathbf{T}_{22}(\lambda)$ is a $(2r-1)\times(2r-1)$ block matrix and $m_1^{\pm}(\lambda)$, $c_1^{\pm}(\lambda)$ are scalar functions.

The reduction of inverse scattering problem to the Riemann-Hilbert problem is achieved by the effective construction of fundamental analytical solutions (FAS), like any other fundamental analytical solution of L, linearly dependent on Jost solutions:

$$\xi(x,t,\lambda) = \psi(x,t,\lambda)\exp(i\lambda Jx), \tag{0.38}$$
$$\varphi(x,t,\lambda) = \phi(x,t,\lambda)\exp(i\lambda Jx). \tag{0.39}$$

Suppose we know the solution to the Riemann-Hilbert problem $\xi_0^{\pm}(x,t,\lambda)$ which has simple poles at the points $\lambda_j^{\pm}\in\mathbb{C}_{\pm}$, $j\neq k$. The dressing method consists in constructing new FAS $\xi^{\pm}(x,t,\lambda)$ which are related to $\xi_0^{\pm}(x,t,\lambda)$ by the dressing factor:

$$\xi^{\pm}(x,t,\lambda) = u(x,t,\lambda)\,\xi_0^{\pm}(x,t,\lambda), \quad \lambda\in\mathbb{R}. \tag{0.40}$$

where $u(x,t,\lambda)$ belongs to the orthogonal group and has poles at the points $\lambda=\lambda_k^{\pm}\neq\lambda_j^{\pm}$ which allows us to choose the following form for the dressing factor:

$$u(x,t,\lambda) = \exp\left(\ln\left(c_k(\lambda)\right)\left(P_k-\bar{P}_k\right)\right). \tag{0.41}$$
$$c_k(\lambda) = \frac{\lambda-\lambda_k^{+}}{\lambda-\lambda_k^{-}}, \tag{0.42}$$
$$\bar{P}_k = S_0P_k^TS_0. \tag{0.43}$$

Chosing the rank 1 prjector P_k such that $P_k\bar{P}_k=0$, we will get:

$$u(x,t,\lambda) = \mathbf{1}+\left(c_k(\lambda)-1\right)P_k+\left(\frac{1}{c_k(\lambda)}-1\right)\bar{P}_k \tag{0.44}$$
$$u^{-1}(x,t,\lambda) = \mathbf{1}+\left(\frac{1}{c_k(\lambda)}-1\right)P_k+\left(c_k(\lambda)-1\right)\bar{P}_k \tag{0.45}$$

The dressing factor and its inverse must satisfy the following equations:

$$i\frac{du}{dx}+\left(Q(x,t)-\lambda J\right)u(x,t,\lambda)-u(x,t,\lambda)\left(Q_0(x,t)-\lambda J\right) = 0, \tag{0.46}$$
$$i\frac{du^{-1}}{dx}+\left(Q_0(x,t)-\lambda J\right)u^{-1}(x,t,\lambda)-u^{-1}(x,t,\lambda)\left(Q(x,t)-\lambda J\right) = 0. \tag{0.47}$$

Taking into account that $u(x,t,\lambda)$ and $u^{-1}(x,t,\lambda)$ have poles at$\lambda=\lambda_k^{\pm}$ the rezidues must be zero, leading fot the following equation fot the projectors P_k şi $\bar{P}_k$:

$$i\frac{dP_k}{dx}+\left(Q(x,t)-\lambda_k^{-}J\right)P_k(x,t)-P_k(x,t)\left(Q_0(x,t)-\lambda_k^{-}J\right) = 0 \tag{0.48}$$
$$i\frac{dP_k}{dx}+\left(Q(x,t)-\lambda_k^{+}J\right)P_k(x,t)-P_k(x,t)\left(Q_0(x,t)-\lambda_k^{+}J\right) = 0 \tag{0.49}$$

Taking the limits of the equations (0.46)-(0.47) for $\lambda \to \infty$ it results the new potential $Q(x,t)$:

$$Q(x,t) - Q_0(x,t) = -\left(\lambda_k^+ - \lambda_k^{+-}\right)\left[J, P_k - \bar{P}_k\right] \tag{0.50}$$

where the projectors P_k can have the following form:

$$P_k = \frac{|n_k\rangle\langle m_k|}{\langle m_k|n_k\rangle}. \tag{0.51}$$

For $Q_0 = 0$ we have $|n_k\rangle = e^{(z_k - i\phi_k)J}|n_{k0}\rangle$ where:

$$\lambda_k^\pm = \mu_k \pm i\nu_k, \qquad z_k = \nu_k(x + 2\mu_k t), \qquad \phi_k = \mu_k x + (\mu_k^2 - \nu_k^2)t, \tag{0.52}$$

and $|n_{k0}\rangle$, $\langle m_{k0}|$ are the polarization vectors constrained by $\langle m_{k0}|S_0|n_{k0}\rangle = 0$.

The soliton solution for the KS model is parameterized by the eigenvalues $\lambda_k^\pm$ and by the polarization vectors $|n_{k0}\rangle$ si $\langle m_{k0}|$, $\langle m_{k0}| = |n_{k0}\rangle^\dagger$. If we introduce:

$$|n_{k0}\rangle = (n_{k0,1}, \nu_{k0}, \bar{n}_{k0,1})^T, \quad \langle m_{k0}|S_0|n_{k0}\rangle = 2n_{k0,1}\bar{n}_{k0,1} - \nu_{k0}^T s_0 \nu_{k0} = 0, \tag{0.53}$$

for $\chi_0^\pm(x,t,\lambda) = \exp(-i(\lambda x + \lambda^2 t)J)$ the regular solution of the RHP, for $r = 3$ we have the 3-component KS model and its one-soliton solution $\vec{q}_{1s}$:

$$\vec{q}_{1s}(x,t;z_1,\phi_1) = -\frac{i\sqrt{2}\nu_1 e^{-i(\phi_1)}\left(e^{-z_1}s_0|\vec{\nu}_{01}\rangle + e^{z_1}|\vec{\nu}_{01}^*\rangle\right)}{\cosh(2z_1) + (\vec{\nu}_{01}^\dagger, \vec{\nu}_{01})}.$$

The results obtained can be generalized and extended offering future research and development directions [P3], [P4].

The **sixth part** of the paper contains the conclusions of the dissertation and the **last part** lists the bibliography used and cited during the elaboration of this paper.

References

[1] A. D. Polyanin, Valentin F. Zaitsev. Handbook on nonlinear partial differential equations. Chapman & Hall/CRC. ISBN 1-58488-355-3. 2004.

[2] H. D. I. Abarbanel, M. I. Rabinovich, M. M Sushchik, Introduction to nonlinear dynamics for physicists, World Scientific Publications Co. Pte. Ltd. 1993.

[3] Julien Clinton Sprott, Elegant Chaos, World Scientific Publishing Co Pte Ltd, 2010.

[4] Lorenz EN. Deterministic Nonperiodic Flow. J Atmos Sci. 1963;20:130-41.

[5] J. David Logan, An Introduction to Nonlinear Partial Differential Equations, A JOHN WILEY & SONS, INC., PUBLICATION, 2008.

[6] T. B. Benjamin, J. L Bona, J. J. .Mahony, "Model equations for long waves in nonlinear dispersive systems", Society Philosophical Transactions Mathematical Physical & Engineering Sciences, 1972, Vol. 272.

[7] A.R. Seadawy, A. Sayed, " Traveling wave solutions of the Benjamin-Bona-Mahony water wave equation", Abstract and applied analysis, 2014, Vol. 2014

[8] Durmus Daghan, Orhan Donmez, Adnan Tuna, "Explicit solutions of the nonlinear partial differential equations", Nonlinear analysis. Real world applications, 2010, Vol.11, 2152-2163.

[9] Kudryashov N.A., One method for finding exact solutions of non linear diferential equations. Commun. Nonlinear Sci. Numer. Simul., 2012, 17, 2248–2253.

[10] Vakhnenko V.O., Parkes E.J., Morrison A.J., A Bäcklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation, Chaos Soliton Fract., 2003,17, 683–692.

[11] Kudryashov N.A., A note on the G'/G-expansion method, Appl.Math. Comput., 2010, 217(4), 1755–1758.

[12] Benny D.J., A general theory for interactions between short and long waves, Stud. Appl. Math., 1977, 56(1), 81–94.

[13] E. Tala-Tebue, E.M.E. Zayed, New Jacobi elliptic function solutions, solitons and other solutions for the (2+212 1)-dimensional nonlinear electrical transmission line equation, Eur. Phys. J. Plus 133(8) (2018) 314.

[14] A.M. Wazwaz, The tanh method: solitons and periodic solutions for the Dodd–Bullough–Mikhailov and the Tzitzeica–Dodd–Bullough equations, Chaos Soliton Fract. 25(1) (2005) 55–63.

[15] J.H. He, X.H Wu, Exp-function method for nonlinear wave equations, Chaos Soliton Fract. 30 (3) (2006) 267 700–708.

[16] G. Xu, Z. Li, Exact travelling wave solutions of the Whitham–Broer–Kaup and Broer–Kaup–Kupershmidt equations, Chaos Soliton Fract. 24(2) (2005) 549–556.

[17] W. Malfiet, The tanh method: a tool for solving certain classes of non-linear PDEs, Mathematical methods in the Applied Sciences, 2005, Vol. 28 (17), 2031-2035.

[18] Cimpoiasu, Rodica. Nerve pulse propagation in biological membranes — Solitons and other invariant solutions. International Journal of Biomathematics. Volume 9, issue 5 (2016).

[19] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves.Comm. Pure Appl. Math., 21:467–490, 1968.

[20] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self focusing and one-dimensional self-modulation of waves in nonlinear media. Sov.Phys.-JETP, 34:62–69, 1972.

[21] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. Stud. Appl. Math., 53:249–315, 1974.

[22] P. P. Kulish and E. K. Sklyanin(1981). O(N)-invariant nonlinear Schrodinger equation - a new completely integrable system, Phys. Lett. 84A, 349–352.

[23] Gerdjikov, V.S.:Basic aspects of soliton theory. In: Mladenov, I.M., Hirshfeld, A.C. (eds.) Geometry, Integrability and Quantization, pp. 78–125. Softex, Sofia (2005).

[24] Gerdjikov, V.S., Kostov, N.A., Valchev, T.I.: Bose-Einstein condensates with F = 1 and F =2: reductions and soliton interactions of multi-component NLS models. In: Proceedings of SPIE, vol. 7501, 7501W (2009).

[25] Ivanov, R.: On the dressing method for the generalised Zakharov-Shabat system. Nucl. Phys.B 694, 509–524 (2004).

[26] Zakharov, V.E., Manakov, S.V., Novikov, S.P., Pitaevskii, L.I.: Theory of solitons. The inverse scattering method, Plenum, N.Y. (1984).

[27] E. V. Doktorov, S. B. Leble. A dressing method in mathematical physics. Mathematical physics study 28. Springer Verlag, Berlin (2007).

Author's contribution papers

ISI1. R. Cimpoiasu, A. Pauna, Complementary wave solutions for the long-short wave resonance model via the extended trial equation method and the generalized Kudryashov method, Open Physics 16 (1), 2018, pp. 419-426, 2018. https://doi.org/10.1515/phys-2018-0057.

ISI2. R. Cimpoiasu, R. Constantinescu, A. Streche Pauna. Solutions of the Bullough-Dodd model of scalar field through Jacobi type equations, Symmetry 13(8), 2021, 1529. https://doi.org/10.3390/sym13081529.

P1. R. Constantinescu, , C.Ionescu , A. Florian, A. Streche (Păuna), Power law method for finding soliton solutions of the 2+1 Ricci flow model, "Proceedings of 9th Mathematical Physics Meeting", Institute of Physics Belgrade, ISBN: 978-86-82441-48-9 (2018), pp 135-146. http://mphys9.ipb.ac.rs/proceedings.html

P2. R. Constantinescu, F. Iacobescu, A. Pauna, Nonlinear mathematical models for physical phenomena, AIP Conference Proceedings 2075 (2019), 100005.

P3. R. Constantinescu, C. Ionescu, A. Pauna, A reduction method for solving nonlinear PDEs, Physics AUC – vol. 30 (part II) 2020, 158-165.

P4. A. Florian, V. S. Gerdjikov, A. Streche-Pauna, On generalized Kulish-Sklyanin models, Physics AUC, vol. 30 (part II) 2020, 175-195.

P5. A. Streche-Pauna, A. Florian, V. S. Gerdjikov, On the Spectral Properties of Lax Operators Related to BD.I Symmetric Spaces, Chapter in "Advanced Computing in Industrial Mathematics", BGSIAM 2018: Studies in Computational Intelligence, Springer vol 961, 2021, pp 345-358. https://doi.org/10.1007/978-3-030-71616-5_31R.

P6. R. Cimpoiasu, R. Constantinescu, G. Florian, A. Streche (Pauna), Direct methods in finding travelling wave solutions for nonlinear evolutionary phenomena, Proceedings of 2nd CONFERENCE ON NONLINEARITY, 18—22.10.2021, Belgrade, Serbia (in press).

C1. Carmen Ionescu, Mihai Stoicescu, Alina Streche, A special case of Chua system. Chaos and regular behavior, Poster presentation in "The Joint Meeting on Quantum Fields and Nonlinear Phenomena". 09-13 March 2016, Sinaia, Romania.

http://cis01.central.ucv.ro/physics/en/workshop_Sinaia_2016/Contributions.pdf

C2. R.Cimpoiasu, R.Constantinescu, M.A. Streche, Chaos and symmetries in mathematical neural flow models, BELBI Conference, Belgrade, June 2016 (prezentare)

http://alas.matf.bg.ac.rs/~websites/bioinfo/wp-content/uploads/2015/11/program.Friday.pdf

C3. R. Constantinescu, C.Ionescu , A. Florian, A. Streche (Păuna), Methods for solving solitonic equation. Examples. Poster in "The Joint Meeting on Quantum Fields and Nonlinear Phenomena", Sinaia, April, 2018.

http://cis01.central.ucv.ro/physics/en/workshop_Sinaia_2018/program.html

C4. A. Florian, J. Ivic, A. Streche (Păuna), M. Stoicescu, Reduction method for reaction-diffusion equations from biology, Belgrade Bio-Informatics Conference. June, 2018, (prezentare).

http://belbi2018.matf.bg.ac.rs/wp-content/uploads/2020/09/FridayProgramBelBi2018_2.pdf

C5. A. Streche (Păuna), Reduction method for finding traveling wave solutions, Prezentare la New Bulgarian University, Sofia, August 2018.

http://iaps.institute/reduction-method-for-finding-traveling-wave-solutions/