SUPER-ADDITIVITY IN EXAMPLES

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This note aims to convince the reader of the book *Complements* [*)] about the undeniable presence of super-additivity in plenty of natural situations. Therefore, we will list three types of super-additive norms and metrics:

A. In mathematics;
B. In physics;
C. In everyday experiences.

To talk without formulas, the reader may directly refer to part C.

The usual rule of a triangle, saying that *the shortest route between two points is the straight line*, is one of the most frequently accepted statements, in both theory and practice. In terms of distances, this rule takes the form

$$d(A, B) \leq d(A, C) + d(C, B), \qquad \text{(s.a.)}$$

where $d(A, B)$ means distance between the points A and B etc. By extension to a *metric* $d : X \times X \to \mathbb{R}_+$, where X is an arbitrary non-void set and $\mathbb{R}_+$ is the set of positive real numbers, this condition represents the *sub-additivity* (briefly s.a.) of d.

By *super-additivity* (briefly S.a.) of function d we understand the opposite inequality, namely

$$d(A, B) \geq d(A, C) + d(C, B). \qquad \text{(S.a.)}$$

Because S.a. strongly contradicts our experience with distances as well as our standard educational training, it is important for us to identify as many as possible examples of S.a. functions. This means to find particular sets X and appropriate meanings for the values of d, which may help us to accept that S.a. is a natural phenomenon too.

Differently from distances (and generally s.a. metrics), condition (S.a.) is not possible for each side of a triangle, so we must restrain S.a. metrics to subsets $\mathsf{R} \subset X \times X$, usually preorders on X. Most frequently, we take

$$\mathsf{R} = \mathrm{K}^{=} = \mathrm{K} \cup д,$$

where K is a strict order and д is the equality on X. Function $d : \mathsf{R} \to \mathbb{R}_+$ is a S.a. metric on X if inequality (S.a.) holds at all $(A, C), (C, B) \in \mathsf{R}$.

In real linear spaces, we may derive S.a. metrics from S.a. norms, which are functions of the form $\wr\cdot\wr : \mathsf{R}[0] \to \mathbb{R}_+$, where preorder R is compatible with linearity of X. As usually, $d(A, B) = \wr B - A \wr$ at each $(A, B) \in \mathsf{R}$..

[*)] BĂLAN T. Trandafir, *Complements of Hyperbolic Mathematics – from Super-Additivity to Structural Discreteness*, Ed. Universitaria Craiova, 2016

A. S.a. norms and metrics in Mathematics. We claim that generally, each s.a. metric has a S.a companion.

Example A1. For the first time*), I met S.a. norms and metrics in the algebra $\mathbb{H} = \mathbb{R}^2$ of hyperbolic numbers. Each $z \in \mathbb{H}$ has the form $x + jy$, where the main property of $j = (0, 1)$ is $j^2 = +1$. Like the complex ones, the hyperbolic number $z = x + jy$ has the conjugate $\bar{z} = x - jy$, but function $\wr \cdot \wr$, restrained to $\mathbb{H}_1 = \{ z = x + jy \in \mathbb{H} : x > |y| \}$, of values

$$\wr z \wr = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 - y^2} \; ,$$

is a S.a. norm.

Obviously, $\wr \cdot \wr$ is the S.a. companion of the Euclidean norm $\| \cdot \|$ of the complex plane $\mathbb{C}$.

Example A2. Let (S, d) be a metric space and let $X = \mathbb{R} \times S$ be the world of events that possibly happen in S. Relation

$$\mathrm{K} = \{((t, x), (s, y)) \in X^2 : s - t > d(x, y)\}$$

is a strict order on X, and function $\sigma : \mathrm{K}^= \to \mathbb{R}_+$, of values

$$\sigma((t, x), (s, y)) = \sqrt{(s-t)^2 - d^2(x, y)} \; ,$$

is a S.a. metric. In particular, d may derive from a norm, like in the case of the Euclidean $S = \mathbb{R}^n$, $n \in \mathbb{N}^*$.

If $S = \mathbb{R}$, then functions $\wr \cdot \wr_k : \mathrm{K}^= [(0, 0)] \to \mathbb{R}_+$, $k = 1, 2, 3$, of values

$$\wr (x, y) \wr_1 = \sqrt{x^2 - y^2} \; ,$$

$$\wr (x, y) \wr_2 = x \text{ and}$$

$$\wr (x, y) \wr_3 = x - |y| \; ,$$

are remarkable examples of S.a. norms.

Example A3. If $\Pi = \{(x, y), (u, v)) : x \le u \;\&\; y \le v\}$ be the product order in $X = \mathbb{R}^2$, then function "area", i.e. $\mathcal{A} : \Pi \to \mathbb{R}_+$, of values

$$\mathcal{A}((x, y), (u, v)) = (u - x)(v - y),$$

is super-additive. In addition, $\wr \cdot \wr = \sqrt{\mathcal{A}}$ is a S.a. norm.

Other S.a. norms $\wr \cdot \wr_k : \Pi[(0, 0)] \to \mathbb{R}_+$, $k = 1, 2$, have the values

$$\wr (x, y) \wr_1 = \min \{x, y\}, \text{ respectively}$$

$$\wr (x, y) \wr_2 = (\sqrt{x} + \sqrt{y})^2 .$$

These examples allow easy extensions to arbitrary Cartesian products of S.a. metric spaces.

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*) In 1962, as a student in Mathematics at the West University of Timisoara, I had to study some Clifford Algebras in my license thesis.

Example A4. Let $(X, (\,.\, | \,.\,))$ be an indefinite inner product space over $\mathbb{R}$ or $\mathbb{C}$.. If $(x \mid x) > 0$ and $(y \mid y) > 0$, but $\mathrm{Lin}\{x \mid y\}$ is an indefinite linear subspace of X, then the fundamental inequality, namely

$$|(x \mid y)|^2 > (x \mid x)\,(y \mid y) \qquad \text{(Aczél)}$$

is opposite to the Cauchy-Bunjakowski-Schwartz inequality. Consequently, the indefinite inner products generate S.a. norms on subspaces of X, which resemble a Pontrjagin space.

Example A5. If T is an arbitrary non-void set, we note by $X = \mathcal{F}_{\mathbb{R}}(T)$ the set of all functions $f : T \to \mathbb{R}$. Relation

$$\mathrm{R} = \{(f, g) \in X^2 : \exists\ \inf\{g(t) - f(t) : t \in T\} > 0\} \cup \text{д}$$

is a strict order on X and function $\wr \cdot \wr : \mathrm{R}[0] \to \mathbb{R}_+$, of values

$$\wr f \wr = \inf\{ f(t) : t \in T\}$$

is a S.a. norm on X. Obviously, $\wr \cdot \wr$ is a S.a. peer of the $\sup$ type norm.

Example A6. Let $(M, \mathcal{A}, \mu)$ be a measure space, and $X = \mathcal{L}^p_{\mathbb{R}}(M)$ be the space of all measurable functions on M, for which $\int |f|^p < \infty$. If $p \in [0, 1)$ and R is the usual order on X, i.e.

$$\mathrm{R} = \{(f, g) : f(x) \le g(x),\ \forall x \in [a, b]\},$$

then function $\wr \cdot \wr : \mathrm{R}\,[0] \to \mathbb{R}_+$, of values $\wr f \wr = \left(\int |f|^p\right)^{1/p}$ is a S.a. norm on X. In fact, this is a consequence of the inequality

$$\int fg \ge \left(\int |f|^p\right)^{1/p} \left(\int |f|^q\right)^{1/q}, \qquad \text{(Hölder)}$$

which holds for $0 < p < 1$ and $p^{-1} + q^{-1} = 1$.

Similar results hold for $p < 0$.

The framework of such S.a. norms allows an appropriate study of the duality of the $\mathcal{L}^p_{\mathbb{R}}$ spaces in the case $p < 1$.

Example A7. In the Boolean algebra of propositions, now noted X, we define the relation of implication $A \Rightarrow B$ by "B is true whenever A is true". It is easy to see that $\Rightarrow$ is an order in X and function $d : \Rightarrow \to \mathbb{R}_+$, of values

$$d(A, B) = \mathrm{card}\,[A, B] - 1,$$

where $[A, B] = \{P \in X : A \Rightarrow P \Rightarrow B\}$, is a S.a. metric on X.

To produce new S.a. norms and metrics, we have a lot of possibilities: restrictions, prolongations, symmetric companions, polarity, quantization, etc. In addition, we may use S.a. normed and metric spaces to construct subspaces, over spaces, product and quotient spaces, and generally initial and final structures.

Obviously, several S.a. norms and metrics from above are isomorphic.

B. S.a. norms and metrics in Physics. The presence of S.a. norms and metrics is a common feature of the Relativist and Quantum Physics.

Example B1. (Proper time) Operating with events, i.e. pairs *time – space*, instead of material points, is a far-famed idea of the Einsteinian Relativity. Represented in a system of coordinates, the events have the form

$$e = (t; x, y, z) \in \mathbb{R} \times \mathbb{R}^3 = \mathfrak{E} ,$$

in which t is the moment and (x, y, z) is the place where e holds. Frequently, the events signify emissions or receptions of signals, as well as the simple fact that an observer has the position (x, y, z) at the moment t.

The fundamental principle of Relativity states that the speed of light

$$c = 299\,792\,456 \; m/s$$

is a universal constant and the relative speed v between any two particles, observers etc. obey the inequality $|\, v \,| < c$. Consequently, the quadratic form

$$Q(e) = c^2\, t^2 - (x^2 + y^2 + z^2) ,$$

which is positive at time-like events, leads to the strict order

$$\mathrm{K} = \{(e_1, e_2) \in \mathfrak{E}^2 : \; Q(e_1 - e_2) > 0 \;\&\; t_1 < t_2\},$$

usually called *causality*. Condition $(e_1, e_2) \in \mathrm{K}$ means that e_2 is accessible to an inertial observer that previously lived the event e_1.

Function $\|\cdot\|_t : \mathrm{K}[0] \to \mathbb{R}_+$, of values $\|e\|_t = \sqrt{Q(e)}$, is a S.a. norm, called *temporal norm*. The meaning of the number $\|e\|_t$ is *proper time*, i.e. time measured by an inertial observer that is moving between the events 0 and e.

The super-additivity of the proper time was initially qualified as a *paradox* (see twins story, clocks paradox etc.), but finally it was verified in practice. Perhaps scientists wouldn't have used term *paradox* if they had known more cases of S.a. norms and metrics.

Example B2. (Quantized norms and metrics) In Quantum Physics, there exist lower bounds for the values of the physical quantities, called *quanta*. The former (~ 1900), quanta of action, discovered by Max Planck, is

$$h = 6.6 \times 10^{-34} \; joule \cdot \mathrm{sec} . \qquad \text{(Planck)}$$

In particular, the shortest measurable length, called *Planck length*, is

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} ,$$

where $\hbar = h / 2\pi$ and G is the gravitational constant.

Quantum Physics reveals a lot of phenomena inexplicable by the classical theories. For example, the quantization of a measurement rejects the usual norms and metrics, since the existence of a lower bound contradicts sub-additivity. On the other hand, all S.a. norms and metrics naturally support the process of quantization. Thus, if $\wr\cdot\wr : \mathrm{R}[0] \to \mathbb{R}_+$ is a S.a. norm in the linear space X, ordered by R, then for arbitrary (generic) $\hbar>0$, the *quantized* function $\wr\cdot\wr_{\hbar}:\mathrm{R}\,[0]\to\mathbb{R}_+$, of values

$$\wr x\wr_{\hbar}=\begin{cases}0 & \text{if } \wr x\wr\le\hbar\\ \wr x\wr & \text{if } \wr x\wr>\hbar\end{cases}$$

is a S.a. (semi-) norm too.

Similarly, we may replace R by

$$\mathrm{R}_{\hbar}=\{(x,y)\in\mathrm{R}:\wr y-x\wr>\hbar\}\cup\text{д},$$

to obtain a threshold $\hbar$ of the restriction $\wr\cdot\wr\big|_{\mathrm{R}_{\hbar}}$.

Example B3. (Heisenberg's uncertainty principle) Besides the individual quantization of the physical quantities, the measurement of each parameter of a particle perturbs the values of the other parameters. Consequently, the measurements at quantum scale obey the so called *principle of uncertainty*. In the particular case of x = *position* and p = *impulse*, this principle gives

$$|\Delta x|\,|\Delta p|>\hbar, \qquad \text{(Heisenberg)}$$

where Δx and Δp represent the errors of measurement of x and p.

Alternatively, the errors of measurement of x and p involve the S.a. norm

$$\wr(|\Delta x|,|\Delta p|)\wr_{\hbar}=\begin{cases}0 & \text{if } |\Delta x|\cdot|\Delta p|\le\hbar\\ \sqrt{|\Delta x|\cdot|\Delta p|} & \text{if } |\Delta x|\cdot|\Delta p|>\hbar\end{cases},$$

which is a quantization of the S.a. norm $\wr\cdot\wr$ from Example A3.

C. Super-additivity in everyday experiences. It's amazing to realize in how many situations and how easy we neglect recognizing super-additivity. Again, the opposition to "*the shortest route on a straight line*" may explain this habitude.

Example C1. (Synergy) It is easy to see that the power (value, efficiency etc.) of a group of persons (or other living beings) is greater than the sum of powers of the members. This is the reason why people live in groups, parties and societies, animals make herds etc.

Example C2. (Proofs) At any level, we may immediately remark that Mathematics (and any other science) consists of sentences of the form

"*If H = hypothesis, then C = conclusion*", (P)

meaning Propositions, Theorems, Problems etc. Understanding (accepting, agreeing with) such assertions usually encounters some *difficulty*. Therefore, to convince that (P) is true, we have to give proofs, which insert at least one intermediate fact, say I, between H and C. The profit of the proof is that the sum of difficulties of the assertions "*If H, then I*" and "*If I, then C*" is less than the initial difficulty of justifying (P).

To conclude, the difficulty of a logical implication is super-additive.

Example C3. (Arts) Like in sciences, we may recognize super-additivity in arts too. Each artist aims to communicate ideas, feelings, advices etc. His piece of art (novel, song, painting etc.) represents the intermediate factor at hand that may reduce the people's difficulty to perceive and accept him.
Example C4. (Using tools) In practice, we often need additional devices to perform an operation. To paraphrase a famous story, let us suppose we see an apple in the tree, but it doesn't fall itself. To reduce the difficulty of getting the fruit, we may look for a pole (stone etc.), then use it to shake down the apple.

Examples like the *Trojan horse* are very frequent in history.
Example C5. (Getting help) To perform hard works we often ask for help. In this round-about way we save energy and time.

Unfortunately, the help is sometimes done illegally or unfairly (by tricks, traitors etc.).
Example C6. (Games) It is widely accepted that many activities in our life represent games. Aiming to win, each player follows some strategy and tries to find the easiest way that leads to victory.
Example C7. (Intelligence) To survive, each creature has to do something, which makes supportable its existence. Hereditarily, it acquires some ways of behavior, but new schemes may result by experience. In the last case, we speak of *intelligence*, i.e. a capacity of the beings to diminish the difficulties of their existence. This process involves the super-additivity of the amount of difficulties realized by intermediate actions. Most frequently, we may highlight super-additivity it in terms of *time*.

Thus, we may conclude that super-additivity is present in plenty of situations. In fact, even if at the beginning I was wandering where to see super-additive phenomena, now I am asking myself about their universality. According to the last Chapter of the book *Complements*, super-additivity is a sign of *structural discreteness*. Encouraged by the general duality between continuity and discreteness, we may assume that sub- and super-additive phenomena equally happen overall in the world.

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